Vaje MAT VSP, 19.11.2020 – ODVOD

NALOGA 32.
Uporabi definicijo odvoda funkcije f v točki x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

in s pomočjo le-te izračunaj odvod funkcije $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{\cancel{f}(x+h) - \cancel{f}(x)}{h} =$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x+h) = 2x$$

Pravila za odvajanje:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(cf)'(x) = cf'(x)$
- $(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$; $g(x) \neq 0$
- $\big(\underbrace{f}_{\text{zunanja}}(\underbrace{g(x)}_{\text{notranja}})\big)' = f'(g(x)) \cdot g'(x)$

Tabela odvodov elementarnih funkcij

$f(x)$	$f'(x)$
c	0
x^m	mx^{m-1}
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

NALOGA 33.

S pomočjo pravil za odvajanje izračunaj odvode naslednjih funkcij spremenljivke x:

a. $x^3 + 5x^2 - 3x + 1$,

b. $\dfrac{2x^2 - 3}{5x + 1}$,

c. $\dfrac{x}{\sqrt{x^2 + x}}$,

d. e^{x^2},

e. $\sin(5x)$,

f. $\tan(x)$,

g. $\dfrac{\sin(x) + \cos(x)}{\sin(x) - \cos(x)}$,

h. $x^3 \log(-3x)$,

i. $\log(\log(x))$,

j. $\arcsin(\cos(x))$,

k. $\dfrac{5^x}{3^{x^2}}$,

l. $\sin^2(-3x)$,

m. $\tan\left(\dfrac{1}{2x^2}\right)$.

a) $(x^3+5x^2-3x+1)' = 3x^2+10x-3$

b) $\left(\dfrac{2x^2-3}{5x+1}\right)' = \dfrac{4x(5x+1)-(2x^2-3)\cdot 5}{(5x+1)^2} =$

$= \dfrac{20x^2+4x-10x^2+15}{(5x+1)^2} =$

$= \dfrac{10x^2+4x+15}{(5x+1)^2}$

c) $\left(\dfrac{x}{\sqrt{x^2+x}}\right)' =$

$(\sqrt[2]{x^2+x})' = \left((x^2+x)^{\frac{1}{2}}\right)' = \frac{1}{2}(x^2+x)^{-\frac{1}{2}} \cdot (2x+1) =$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$g(x) = x^2+x$

$f(g(x)) = \sqrt{x^2+x}$

$\sqrt[m]{x^n} = x^{\frac{n}{m}}$

$= \dfrac{1\cdot(2x+1)}{2(x^2+x)^{\frac{1}{2}}} = \dfrac{2x+1}{2\sqrt{x^2+x}}$

$$= \frac{1\cdot\sqrt{x^2+x} - x \cdot \frac{2x+1}{2\sqrt{x^2+x}}}{x^2+x} =$$

$$= \frac{\left[\frac{2(x^2+x)-(2x^2+x)}{2\sqrt{x^2+x}}\right]}{\frac{x^2+x}{1}} =$$

$$= \frac{\cancel{2x^2}+2x-\cancel{2x^2}-x}{2(x^2+x)^{\frac{1}{2}}\cdot(x^2+x)^1} = \frac{x}{2\ (x^2+x)^{\frac{3}{2}}}$$

d) $(e^{x^2})' = e^{x^2}\cdot 2x$

$f(x) = e^x$
$g(x) = x^2$
$f(g(x)) = e^{x^2}$

e) $(\sin(5x))' = \cos(5x)\cdot 5 = 5\cos(5x)$

f) $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' =$

$$= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

g) $\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)' =$

$$= \frac{(\cos x-\sin x)\cdot(\sin x-\cos x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{\cancel{\cos x\cdot\sin x}-\cos^2 x-\sin^2 x+\cancel{\sin x\cos x}-\sin^2 x-\cancel{2\sin x\cos x}-\cos^2 x}{\sin^2 x-2\sin x\cos x+\cos^2 x}$$

$$= \frac{-(\overbrace{\cos^2 x+\sin^2 x}^{1}+\overbrace{\sin^2 x+\cos^2 x}^{1})}{1-\sin(2x)} =$$

$$= \frac{-2}{1-\sin(2x)} = \frac{2}{\sin(2x)-1}$$

h) $(\underbrace{x^3}_{f}\cdot\underbrace{\log(-3x)}_{g})' = 3x^2\cdot\log(-3x)+x^3\cdot\frac{1}{-\cancel{3}x}\cdot(-\cancel{3})$

$$= 3x^2\log(-3x)+x^2$$

$$= x^2(3\log(-3x)+1)$$

i) $(\log(\log x))' = \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}$

j) $(\arcsin(\cos x))' = \frac{1}{\sqrt{1-\cos^2 x}}\cdot(-\sin x) =$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

k) $\left(\frac{5^x}{3^{x^2}}\right)' = \frac{5^x \cdot \log 5 \cdot 3^{x^2} - 5^x \cdot 3^{x^2} \cdot \log 3 \cdot 2x}{\left(3^{x^2}\right)^2} =$

$= \frac{5^x \cdot \cancel{3^{x^2}} \left(\log 5 - 2x \log 3\right)}{\left(3^{x^2}\right)^{\cancel{2}}} =$

$= \frac{5^x \left(\log 5 - 2x \log 3\right)}{3^{x^2}}$

ℓ) $\left(\sin^2(-3x)\right)' = 2\sin(-3x) \cdot \left(\sin(-3x)\right)'$

$\|$

$\left(\left(\sin(-3x)\right)^2\right)' = 2\sin(-3x) \cdot \cos(-3x) \cdot (-3) =$

$= -2\sin(3x) \cdot \cos(3x) \cdot (-3) =$

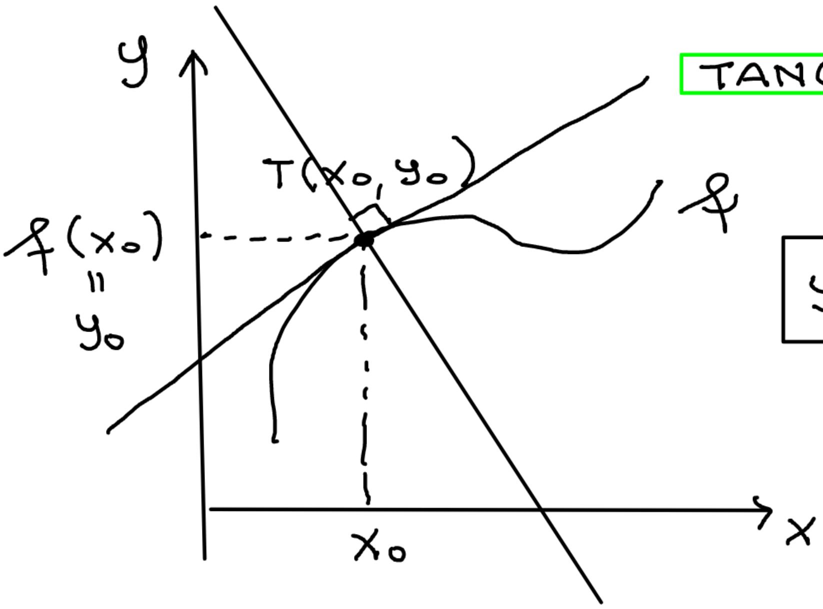
$= 6\sin(3x) \cdot \cos(3x) =$

$2\sin x \cdot \cos x = \sin(2x)$

$= 3 \cdot \underbrace{2\sin(3x) \cdot \cos(3x)}$

$= 3 \cdot \sin(2 \cdot 3x) = 3\sin(6x)$

TANGENTE IN NORMALE



$\boxed{\text{TANGENTA}}$ na graf funkcije f v T:

$k_T = f'(x_0)$

$$\boxed{y - y_0 = f'(x_0) \cdot (x - x_0)}$$

$\boxed{\text{NORMALA}}$ na graf funkcije f v T

$k_N = -\frac{1}{k_T} = -\frac{1}{f'(x_0)}$

$$\boxed{y - y_0 = -\frac{1}{f'(x_0)} \cdot (x - x_0)}$$

NALOGA 35. *OR*

Funkcija f ima predpis

$$f(x) = x^3 - 2x^2 + 3.$$

Poišči enačbo tangente na graf te funkcije v točki $(1, f(1))$ ter enačbo normale na graf v točki $(2, f(2))$. (Normala je premica, ki je pravokotna na tangento v dani točki.) V kateri točki se ti dve premici sekata?

$f(x) = x^3 - 2x^2 + 3$

- tangento na graf f v točki

$T(1, f(1)) = T(\overset{x_0}{1}, \overset{y_0}{2})$

$f(1) = 1^3 - 2 \cdot 1^2 + 3 = 2$

$k_T = f'(1)$

$$f'(x) = 3x^2 - 4x$$

$$k_T = f'(1) = 3 \cdot 1^2 - 4 \cdot 1 = -1$$

$$y - y_0 = k_T(x - x_0)$$

$$y - 2 = -1(x - 1)$$

$$\boxed{y = -x + 3}$$

- normala na graf funkcije f v

$$T(2, f(2)) = T(\overset{x_0}{2}, \overset{y_0}{3})$$

$$f(2) = 2^3 - 2 \cdot 2^2 + 3 = 3$$

$$k_N = -\frac{1}{f'(2)} = -\frac{1}{4}$$

$$f'(2) = 3 \cdot 2^2 - 4 \cdot 2 = 4$$

$$y - y_0 = -\frac{1}{4}(x - x_0)$$

$$y - 3 = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + \frac{2}{4} + 3$$

$$\boxed{y = -\frac{1}{4}x + \frac{7}{2}}$$

PRESEČIŠČE: $\quad y = y$

$$-x + 3 = -\frac{1}{4}x + \frac{7}{2} \quad /\cdot 4$$

$$-4x + 12 = -x + 14$$

$$-3x = 2 \quad /:(-3)$$

$$x = -\frac{2}{3}$$

$$y = -x + 3$$

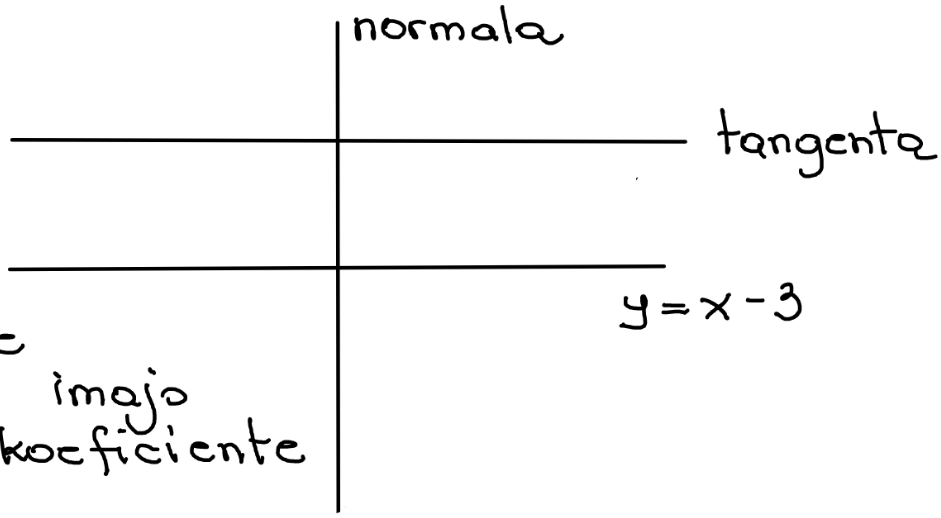
$$y = \frac{2}{3} + 3 = \frac{11}{3}$$

$$\boxed{P\left(-\frac{2}{3}, \frac{11}{3}\right)}$$

NALOGA 37. *OR*

Poišči tisto normalo na graf funkcije $y = x\log(x)$, ki je pravokotna na premico z enačbo $y = x - 3$.

$$y = x \log x = f(x)$$



$$y = x - 3$$

$$\left.\begin{array}{l} k = 1 \\ \\ k_T = 1 \end{array}\right\} \text{vzporedne premice imajo enake koeficiente}$$

$$\|$$

$$f'(x) = 1$$

$$f'(x) = 1 \cdot \log x + x \cdot \frac{1}{x} = $$
$$= 1 \cdot \log x + 1$$

$$\log x + \cancel{1} = \cancel{1}$$

$$\log x = 0$$

$$\log_e x = 0$$

$$e^0 = x$$

$$\underline{x = 1}$$

Iščemo enačbo normale na graf f

v točki $T(1, \underset{\substack{\| \\ 1 \cdot \log_e 1 = 0}}{f(1)}) = T(\overset{x_0}{1}, \overset{y_0}{0})$ $\quad e^{0} = 1$

$$k_N = -\frac{1}{k_T} = -\frac{1}{1} = -1$$

$$y - y_0 = k_N (x - x_0)$$

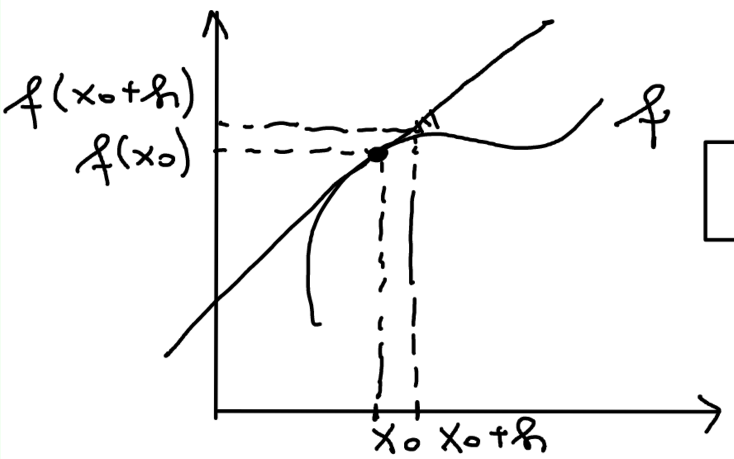
$$y - 0 = -1(x - 1)$$

$$\boxed{y = -x + 1}$$

NALOGA 40.
Z uporabo totalnega diferenciala določi približno vrednost spodnjih izrazov:

a. $\arctan(0.03)$,
b. $\sqrt{4.1}$,
c. $\sqrt[3]{25}$,
d. $\log(0.9)$.

b) $\sqrt{4.1}$



$$\boxed{f(x_0 + h) \doteq f(x_0) + h f'(x_0)}$$

x_0 … točka, v kateri poznam vrednost funkcije f in f'

$x_0 + h$ … točka, v kateri bi radi izračunali približno vrednost funkcije f

$$\sqrt{4.1} = \underset{x_0+h}{\sqrt{4+0.1}} \doteq \sqrt{4} + 0.1 \cdot \frac{1}{4} = 2 + \overset{0.025}{\widetilde{0.1 \cdot 0.25}} = 2{,}025$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$x_0 = 4$

$x_0 + h = 4.1$

$4 + h = 4.1 \Rightarrow h = 4.1 - 4 = 0.1$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

c) $$\sqrt[3]{25} = \sqrt[3]{27} - 2 \cdot \frac{1}{27} = 3 - \frac{2}{27} = \frac{81-2}{27} = \frac{79}{27} =$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \qquad \doteq 2{,}\overline{925}$$

$x_0 = 27$

$h = -2$

$x_0 + h = 25$

$27 + h = 25$

$h = 25 - 27$

$h = -2$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

$$f'(x) = \frac{1}{3} x^{\overset{\frac{1}{3}-1}{\overset{\shortparallel}{-\frac{2}{3}}}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(x_0) = f'(27) = \frac{1}{3\sqrt[3]{27^2}} = \frac{1}{3 \cdot \sqrt[3]{27} \cdot \sqrt[3]{27}} = \frac{1}{27}$$

$$\frac{1}{3\sqrt[3]{27^2}} = \frac{1}{3 \cdot \sqrt[3]{27 \cdot 27}} = \frac{1}{3 \cdot \underbrace{\sqrt[3]{27}}_{3} \cdot \underbrace{\sqrt[3]{27}}_{3}} = \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{27}$$