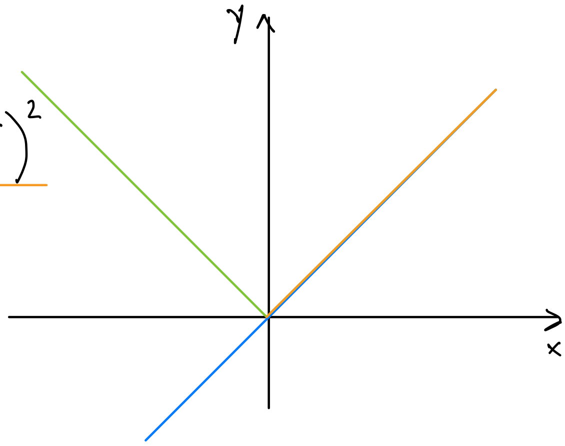
Matematika VSP, vaje, 20. 11. 2020 (funkcije, odvod)

NALOGA 28.
Ali predpisi x, $\sqrt{x^2}$ ter $(\sqrt{x})^2$ predstavljajo iste funkcije?



$f_1(x) = x$, $f_2(x) = \sqrt{x^2}$, $f_3(x) = (\sqrt{x})^2$

$f_2(x) = |x|$

Torej $f_1 \neq f_2$.

Kaj so definicijska območja f_1, f_2, f_3?

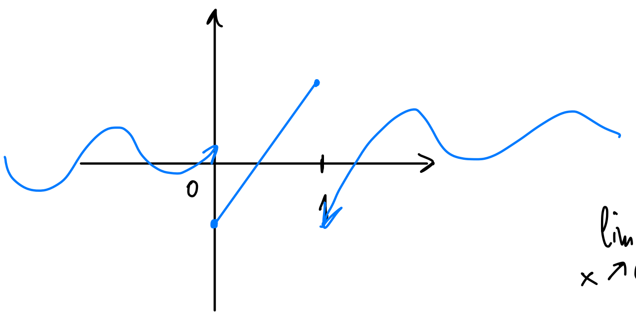
$D_{f_1} = \mathbb{R}$, $D_{f_2} = \mathbb{R}$, $D_{f_3} = [0, \infty)$ (saj je $\sqrt{x}$ definiran le za $x \geq 0$)

(saj je $x^2 \geq 0$ za $x \in \mathbb{R}$)

Torej tudi $f_3 \neq f_1$, $f_3 \neq f_2$.

NALOGA 31.
Določi konstanto a in b tako, da bo f zvezna funkcija.

$$f(x) = \begin{cases} \frac{\sin(3x)(x-2)}{x}, & x < 0 \\ ax + b, & 0 \leq x \leq 1, \\ 2e^{x-1} - \cos(\pi x). & x > 1 \end{cases}$$



Da bo f zvezna v $x = 0$, mora veljati:

$$\lim_{x \nearrow 0} \frac{\sin(3x) \cdot (x-2)}{x} = \lim_{x \searrow 0} (ax+b) = b$$

$$\| $$

$(0-2) \cdot 3 = -6$, torej $b = -6$.

S predavanj:

$$\lim_{x \to 0} \frac{\sin(kx)}{x} = k.$$

Da bo f zvezna tudi v $x = 1$, mora veljati:

$$a + b = \lim_{x \nearrow 1} (ax+b) = \lim_{x \searrow 1} \left(2e^{x-1} - \cos(\pi x)\right) =$$

$$= 2e^{1-1} - \cos(\pi) = 2 - (-1) = 3$$

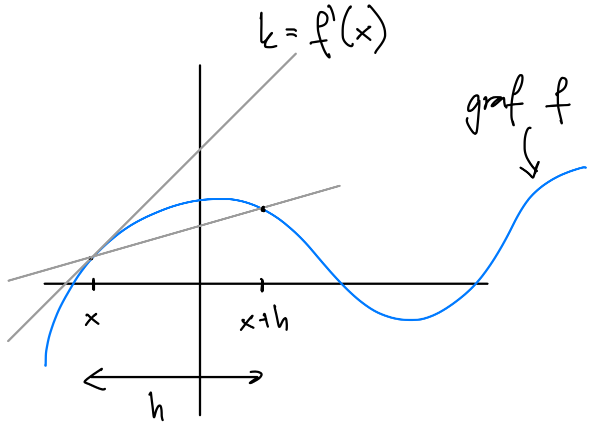
$a + b = 3$, $a - 6 = 3$, $a = 9$.

f bo zvezna za $a = 9$ in $b = -6$.

NALOGA 32.
Uporabi definicijo odvoda funkcije f v točki x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

in s pomočjo le-te izračunaj odvod funkcije $f(x) = x^2$.



$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} =$$

$$= \lim_{h \to 0} \left(\frac{2x\cancel{h}}{\cancel{h}} + \frac{h^{\cancel{2}}}{\cancel{h}} \right) = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x.$$

NALOGA 33.
S pomočjo pravil za odvajanje izračunaj odvode naslednjih funkcij spremenljivke x:

a. $x^3 + 5x^2 - 3x + 1$,
b. $\dfrac{2x^2-3}{5x+1}$,
c. $\dfrac{x}{\sqrt{x^2+x}}$,
d. e^{x^2},
e. $\sin(5x)$,
f. $\tan(x)$,
g. $\dfrac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)}$,
h. $x^3 \log(-3x)$,
i. $\log(\log(x))$,
j. $\arcsin(\cos(x))$,
k. $\dfrac{5^x}{3^{x^2}}$,
l. $\sin^2(-3x)$,
m. $\tan\left(\dfrac{1}{2x^2}\right)$.

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$f_1(x)+f_2(x)$	$f_1'(x)+f_2'(x)$
$C\cdot f(x)$	$C\cdot f'(x)$
$\dfrac{f_1(x)}{f_2(x)}$	$\dfrac{f_1'\cdot f_2 - f_1\cdot f_2'}{(f_2)^2}$
$f_1(f_2(x))$	$f_1'(f_2(x))\cdot f_2'(x)$
$\sin x$	$\cos x$

NALOGA 34.
Poišči odvode naslednjih funkcij spremenljivke x:

a. $\dfrac{x^3-x^2}{x^4}$,
h. $\dfrac{1+\log(x)}{1-\log(x)}$,

33.(a) $\left(x^3+5x^2-3x+1\right)' = \frac{d}{dx}\left(x^3+5x^2-3x+1\right)$

$= 3x^2 + 5\cdot 2x - 3\cdot 1 + 0 = 3x^2 + 10x - 3.$

(b) $\left(\dfrac{2x^2-3}{5x+1}\right)' = \dfrac{2\cdot 2x\cdot(5x+1) - (2x^2-3)\cdot 5}{(5x+1)^2} = \dfrac{10x^2+4x+15}{(5x+1)^2}$.

(e) $\left(\sin(5x)\right)' = \cos(5x)\cdot 5 = 5\cos(5x)$.

$f_1(x) = \sin(x)$,
$f_2(x) = 5x$

(c) $\left(\dfrac{x}{\sqrt{x^2+x}}\right)' = \dfrac{1\cdot\sqrt{x^2+x} - x\cdot\dfrac{1}{2\sqrt{x^2+x}}\cdot(2x+1)}{\left(\sqrt{x^2+x}\right)^2} =$

$f_1(x) = \sqrt{x} = x^{1/2}$

$f_1'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{x}}$

$$= \frac{\sqrt{x^2+x} - \frac{2x^2+x}{2\sqrt{x^2+x}}}{x^2+x} = \frac{\frac{2(x^2+x)-2x^2-x}{2\sqrt{x^2+x}}}{x^2+x} =$$

$$= \frac{x}{2(x^2+x)\sqrt{x^2+x}}.$$

34. (a) $$\left(\frac{x^3-x^2}{x^4}\right)' = \frac{(3x^2-2x)\cdot x^4 - (x^3-x^2)\cdot 4x^3}{x^8} =$$

$$= \frac{3x^6-2x^5-4x^6+4x^5}{x^8} = -\frac{x^6}{x^8} + \frac{2x^5}{x^8} = -\frac{1}{x^2} + \frac{2}{x^3}.$$

$$\left(\frac{x^3}{x^4} - \frac{x^2}{x^4}\right)' = \left(\frac{1}{x} - \frac{1}{x^2}\right)' = \left(x^{-1} - x^{-2}\right)' = -1\,x^{-1-1} - \left(-2x^{-2-1}\right) =$$

$$= -x^{-2} + 2x^{-3} = -\frac{1}{x^2} + \frac{2}{x^3}.$$

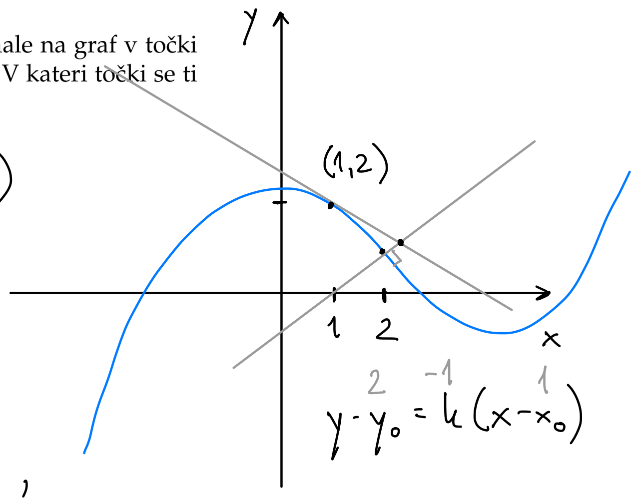
(h) $$\left(\frac{1+\log(x)}{1-\log(x)}\right)' = \frac{\frac{1}{x}\cdot(1-\log x) - (1+\log x)\cdot\left(-\frac{1}{x}\right)}{(1-\log x)^2} =$$

$$(\log x)' = \frac{1}{x}$$

$$= \frac{1}{x}\,\frac{1-\cancel{\log x}+1+\cancel{\log x}}{(1-\log x)^2} = \frac{2}{x\,(1-\log x)^2}.$$

OR

NALOGA 35.
Funkcija f ima predpis
$$f(x) = x^3 - 2x^2 + 3.$$
Poišči enačbo tangente na graf te funkcije v točki $(1, f(1))$ ter enačbo normale na graf v točki $(2, f(2))$. (Normala je premica, ki je pravokotna na tangento v dani točki.) V kateri točki se ti dve premici sekata?



Smerni koef. tangente v $(1, f(1)) = (1,2)$ je $f'(1)$.

$f'(x) = 3x^2 - 2\cdot 2x + 0 = 3x^2 - 4x$,

$f'(1) = -1 = k$ .... $y = kx + n = -x + n$,

$y - y_0 = k(x - x_0)$ — 2, -1, 1

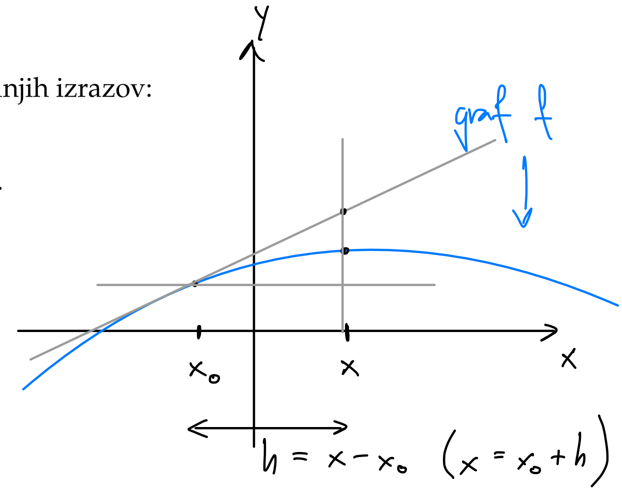
ker gre skozi $(x,y) = (1,2)$, mora veljati še $2 = -1 + n$ ... $n = 3$.

Enačba tangente skozi $(1, f(1))$ je torej $y = -x + 3$.

Še normala skozi $(2, f(2)) = (2,3)$, smerni koef. je $k_n = -\frac{1}{k_t} = -\frac{1}{f'(2)}$.

NALOGA 40.
Z uporabo totalnega diferenciala določi približno vrednost spodnjih izrazov:

a. $\arctan(0.03)$,
b. $\sqrt{4.1}$,
c. $\sqrt[3]{25}$,
d. $\log(0.9)$.



$$f(x) \doteq f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0 + h) \doteq f(x_0) + f'(x_0) \cdot h$$

(b) $f(x) = \sqrt{x}$, $f(4.1) \doteq ?$, $f(4) = \sqrt{4} = 2$,

vzemimo $x_0 = 4$, $x = 4.1$, tj. $x - x_0 = 0.1$.

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(x_0) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} = 0.25.$$

Torej: $\sqrt{4.1} = f(4.1) \doteq f(4) + f'(4) \cdot (4.1 - 4) = 2 + 0.25 \cdot 0.1 =$

$= 2.025$.

(c) $\sqrt[3]{25} \doteq ?$ Vemo $\sqrt[3]{27} = 3$, izberemo $f(x) = \sqrt[3]{x} = x^{1/3}$.

$$f'(x) = \frac{1}{3} x^{\frac{1}{3} - 1} = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}.$$

$$f(27) = 3, \quad f'(27) = \frac{1}{27}.$$

$$f(\underset{x}{25}) \doteq f(\underset{x_0}{27}) + f'(\underset{x_0}{27}) \cdot (\underset{x}{25} - \underset{x_0}{27}) = 3 + \frac{1}{27} \cdot (-2) = 3 - \frac{2}{27} = \frac{79}{27}$$