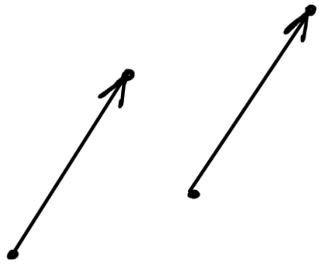
Vaje MAT VSP, 17. 12. 2020

VEKTORJI



$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

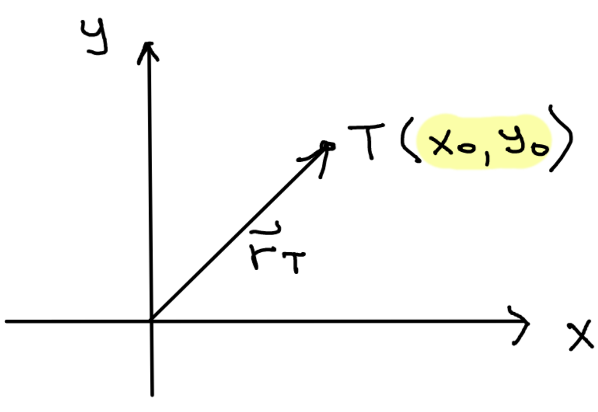
$a_1, a_2, \dots, a_m$ komponente vektorja $\vec{a}$

Seštevanje vektorjev

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix}$$

Množenje s skalarjem

$$\alpha \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_m \end{bmatrix}$$



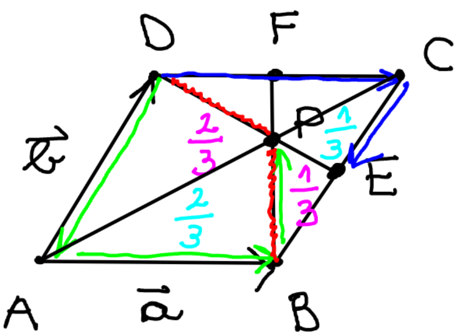
$$\vec{r}_T = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

krajevni vektor točke T

1. V rombu z oglišči $ABCD$ označimo z E točko na razpolovišču stranice BC in z F točko na razpolovišču stranice CD. Naj bo P točka, v kateri se sekata daljici BF in DE.

   (a) Kolikšno je razmerje med dolžinama daljic DP in PE?

   (b) Prepričaj se, da točka P leži na diagonali AC. V kolikšnem razmerju ta točka deli diagonalo?



$|DP| : |PE| = ?$

Če sta $\vec{a}$ in $\vec{b}$ nekolinearna (nevzporedna) vektorja v $\mathbb{R}^2$, potem lahko vsak vektor $\vec{c} \in \mathbb{R}^2$ zapišemo kot linearno kombinacijo vektorjev $\vec{a}$ in $\vec{b}$:

$$\vec{c} = \alpha \vec{a} + \gamma \vec{b}$$

a) $\overrightarrow{DP} = k\,\overrightarrow{DE}$

$\overrightarrow{DP} = k\left(\vec{a} - \frac{1}{2}\vec{b}\right)$

$$\begin{aligned}\overrightarrow{DP} &= -\vec{b} + \vec{a} + \overrightarrow{BP} \\ &= -\vec{b} + \vec{a} + k\,\overrightarrow{BF} \\ &= -\vec{b} + \vec{a} + k\left(\vec{b} - \frac{1}{2}\vec{a}\right)\end{aligned}$$

$\overrightarrow{DP} = \overrightarrow{DP}$

$$k\left(\vec{a} - \frac{1}{2}\vec{b}\right) = -\vec{b} + \vec{a} + k\left(\vec{b} - \frac{1}{2}\vec{a}\right)$$

$$k\left(\vec{a} - \frac{1}{2}\vec{b}\right) - k\left(\vec{b} - \frac{1}{2}\vec{a}\right) = \vec{a} - \vec{b}$$

$$k\left(\vec{a} - \frac{1}{2}\vec{b} - \vec{b} + \frac{1}{2}\vec{a}\right) = \vec{a} - \vec{b}$$

$$k\left(\frac{3}{2}\vec{a} - \frac{3}{2}\vec{b}\right) = \vec{a} - \vec{b}$$

$$k = \frac{\cancel{\vec{a}+\vec{b}}}{\frac{3}{2}(\cancel{\vec{a}+\vec{b}})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\overrightarrow{DP} = \frac{2}{3}\overrightarrow{DE}$$

$$|DP| : |PE| = 2 : 1$$

b) Vektorja sta kolinearna (vzporedna) če je eden večkratnik drugega.

$$\boxed{\overrightarrow{AP} = l\,\overrightarrow{AC}} \qquad l \in \mathbb{R}$$

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

$$\begin{aligned}
\overrightarrow{AP} &= \vec{b} + \overrightarrow{DP} \\
&= \vec{b} + \frac{2}{3}\left(\vec{a} - \frac{1}{2}\vec{b}\right) \\
&= \vec{b} + \frac{2}{3}\vec{a} - \frac{1}{3}\vec{b} \\
&= \frac{2}{3}\vec{a} + \frac{2}{3}\vec{b} \\
&= \frac{2}{3}(\vec{a} + \vec{b})
\end{aligned}$$

$$\overrightarrow{AP} = l\,\overrightarrow{AC}$$

$$\frac{2}{3}(\vec{a}+\vec{b}) = l(\vec{a}+\vec{b})$$

$$l = \frac{2}{3}$$

$$|AP| : |PC| = 2 : 1$$

2. Dana sta vektorja $\vec{a} = [1,1]^T$ in $\vec{b} = [-\sqrt{3},1]^T$. Določi kot med njima in poišči pravokotno projekcijo vektorja $\vec{a}$ na vektor $\vec{b}$.

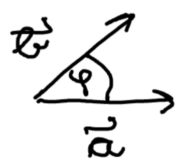
Skalarni produkt vektorjev

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

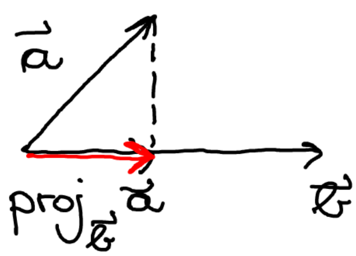
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_m b_m = \text{ŠTEVILO (SKALAR)}$$

- Kot med vektorjema $\vec{a}$ in $\vec{b}$:

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \; ; \quad |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \ldots + a_m^2}$$



- Pravokotna projekcija vektorja $\vec{a}$ na vektor $\vec{b}$:



$$proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \cdot \vec{b}$$

- $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

- $$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{1-\sqrt{3}}{\sqrt{2} \cdot 2} \Rightarrow \varphi = \arccos\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)$$

$$= 105°$$

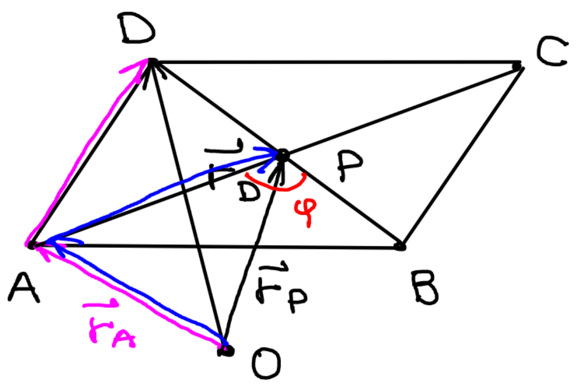
$$\vec{a}\cdot\vec{b} = 1(-\sqrt{3}) + 1\cdot 1 = 1-\sqrt{3}$$

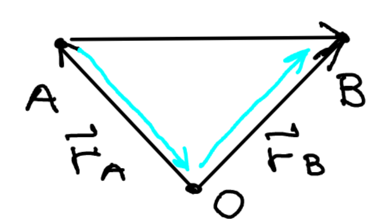
$$|\vec{a}| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$|\vec{b}| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{4} = 2$$

- $$\operatorname{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b} = \frac{1-\sqrt{3}}{4}\cdot\begin{bmatrix}-\sqrt{3}\\1\end{bmatrix}$$

3. Dan je paralelogram $ABCD$ z oglišči $A(-3,-2,0)$, $B(3,-3,1)$, $C(5,0,2)$.
   (a) Določi oglišče D in presečišče diagonal.
   (b) Izračunaj dolžini stranic paralelograma $ABCD$ in kot med njegovima diagonalama.
   (c) Izračunaj ploščino paralelograma $ABCD$.





$$\vec{AB} = -\vec{r}_A + \vec{r}_B$$
$$\vec{AB} = \vec{r}_B - \vec{r}_A$$

a) • $$\vec{r}_D = \vec{r}_A + \vec{AD}$$
$$= \vec{r}_A + \vec{BC}$$
$$= \vec{r}_A + \vec{r}_C - \vec{r}_B$$
$$= \begin{bmatrix}-3\\-2\\0\end{bmatrix} + \begin{bmatrix}5\\0\\2\end{bmatrix} - \begin{bmatrix}3\\-3\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\boxed{D(-1,1,1)}$$

- Diagonali v paralelogramu se razpolavljata.

$$\vec{r}_P = \vec{r}_A + \vec{AP}$$

$$= \vec{r}_A + \frac{1}{2}\vec{AC} =$$

$$= \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \frac{1}{2}\left(\vec{r}_C - \vec{r}_A\right)$$

$$= \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{P(1,-1,1)}$$

b) $A(-3,-2,0)$, $B(3,-3,1)$, $C(5,0,2)$, $D(-1,1,1)$

$|\vec{AB}| = ?$ $\quad |\vec{AD}| = ?$

$$\vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$$

$$|\vec{AB}| = \sqrt{6^2 + (-1)^2 + 1^2} = \underline{\underline{\sqrt{38}}}$$

$$\vec{AD} = \vec{r}_D - \vec{r}_A = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$|\vec{AD}| = \sqrt{2^2 + 3^2 + 1^2} = \underline{\underline{\sqrt{14}}}$$

Računamo kot med vektorjema $\vec{PA}$ in $\vec{PB}$:

$$\cos\varphi = \frac{\vec{PA}\cdot\vec{PB}}{|\vec{PA}||\vec{PB}|} = \frac{-\cancel{6}}{\cancel{3}\sqrt{2}\cdot\cancel{2}\sqrt{2}} = -\frac{1}{2} \Rightarrow \varphi = 120^\circ$$

$$\vec{PA} = \vec{r}_A - \vec{r}_P = \begin{bmatrix}-3\\-2\\0\end{bmatrix} - \begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}-4\\-1\\-1\end{bmatrix}$$

$$\vec{PB} = \vec{r}_B - \vec{r}_P = \begin{bmatrix}3\\-3\\1\end{bmatrix} - \begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\-2\\0\end{bmatrix}$$

$$\vec{PA}\cdot\vec{PB} = (-4)\cdot 2 + (-1)(-2) + (-1)\cdot 0 = -6$$

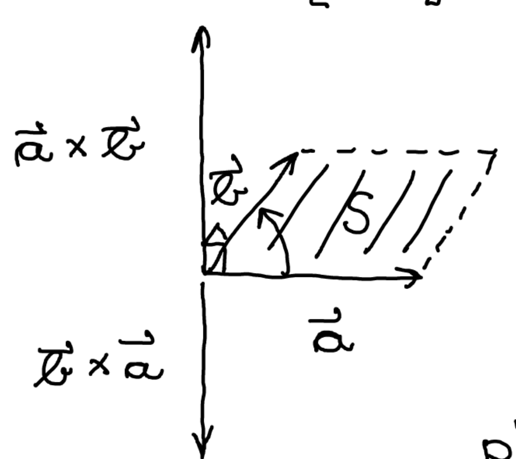
$$|\vec{PA}| = \sqrt{(-4)^2+(-1)^2+(-1)^2} = \sqrt{\underset{9\cdot 2}{\underset{\parallel}{18}}} = 3\sqrt{2}$$

$$|\vec{PB}| = \sqrt{2^2+(-2)^2+0^2} = \sqrt{\underset{4\cdot 2}{\underset{\parallel}{8}}} = 2\sqrt{2}$$

c) Izračunati moramo $S_{\square}$.

Vektorski produkt

$$\vec{a}\times\vec{b} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}a_2b_3 - a_3b_2\\-(a_1b_3 - a_3b_1)\\a_1b_2 - a_2b_1\end{bmatrix}$$



$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$S_{\square} = |\vec{a}\times\vec{b}|$$

ploščina paralelograma napetega na vektorja $\vec{a}$ in $\vec{b}$ je $|\vec{a}\times\vec{b}|$

dolžina vektorja

$$\vec{AB} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix} = \vec{a}$$

$$\vec{AD} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \vec{b}$$

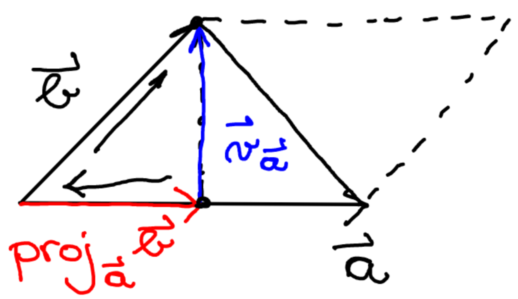
$$\vec{a} \times \vec{b} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)\cdot 1 - 3 \cdot 1 \\ -(6 \cdot 1 - 2 \cdot 1) \\ 6 \cdot 3 - 2(-1) \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 20 \end{bmatrix}$$

$$S_{▱} = |\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (-4)^2 + 20^2} = \underline{\underline{\sqrt{432}}}$$

4. (a) Izračunaj kot med vektorjema $\vec{a} = [2, -2, 4]^T$ in $\vec{b} = [2, 4, -2]^T$.
   (b) Kolikšna je ploščina trikotnika, ki ga ta dva vektorja določata?
   (c) Poišči pravokotno projekcijo vektorja $\vec{b}$ na vektor $\vec{a}$ in še vektor, ki v danem trikotniku predstavlja višino na $\vec{a}$.

a) $$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

b)



ploščina trikotnika, napetega na vektorja $\vec{a}$ in $\vec{b}$:

$$S_{\Delta} = \frac{|\vec{a} \times \vec{b}|}{2}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} (-2)(-2) - 4 \cdot 4 \\ -(2(-2) - 2 \cdot 4) \\ 2 \cdot 4 - 2(-2) \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \\ 12 \end{bmatrix}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-12)^2+12^2+12^2} = \sqrt{3\cdot 12^2} = 12\sqrt{3}$$

$$S_\Delta = \frac{12\sqrt{3}}{2} = \underline{\underline{6\sqrt{3}}}$$

c) • $$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a} = \frac{-12}{24}\begin{bmatrix}2\\-2\\4\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}2\\-2\\4\end{bmatrix} = \underline{\underline{\begin{bmatrix}-1\\1\\-2\end{bmatrix}}}$$

$$\vec{a}\cdot\vec{b} = 2\cdot 2 + (-2)\cdot 4 + 4(-2) = -12$$

$$|\vec{a}| = \sqrt{2^2+(-2)^2+4^2} = \sqrt{24}$$

• $$\vec{v}_{\vec{a}} = -\text{proj}_{\vec{a}}\vec{b} + \vec{b} =$$

$$= -\begin{bmatrix}-1\\1\\-2\end{bmatrix} + \begin{bmatrix}2\\4\\-2\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix} + \begin{bmatrix}2\\4\\-2\end{bmatrix} = \begin{bmatrix}3\\3\\0\end{bmatrix}$$

5. Poišči vektor $\vec{a}$, ki je pravokoten na vektorja $\vec{b} = [4,1,9]^T$ ter $\vec{c} = [-2,2,3]^T$ in ima dolžino 7.

Vektor, ki je pravokoten na vektorja $\vec{b}$ in $\vec{c}$, je vektor $\vec{b}\times\vec{c}$:

$$\vec{b}\times\vec{c} = \begin{bmatrix}4\\1\\9\end{bmatrix} \times \begin{bmatrix}-2\\2\\3\end{bmatrix} = \begin{bmatrix}1\cdot 3 - 2\cdot 9\\ -(4\cdot 3-(-2)\cdot 9)\\ 4\cdot 2-(-2)\cdot 1\end{bmatrix} = \begin{bmatrix}-15\\-30\\10\end{bmatrix}$$

Vektor, ki kaže v smeri vektorja $\vec{b}\times\vec{c}$ in ima dolžino ena:

$$\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} = \frac{1}{|\vec{b}\times\vec{c}|}\cdot\vec{b}\times\vec{c} = \frac{1}{35}\cdot\begin{bmatrix}-15\\-30\\10\end{bmatrix}$$

$$|\vec{b} \times \vec{c}| = \sqrt{(-15)^2 + (-30)^2 + 10^2} = 35$$

Vektor, ki kaže v smeri $\vec{b} \times \vec{c}$ in ima dolžino 7:

$$\vec{a}_1 = 7 \cdot \frac{1}{35} \begin{bmatrix} -15 \\ -30 \\ 10 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -15 \\ -30 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$$

6. Dane so točke $A(1,1,2)$, $B(1,4,-1)$, $C(3,3,2)$ in $D(4,-1,4)$.

   (a) Izračunaj prostornino paralelepipeda, ki je napet na vektorje AB, AC in AD.

   (b) Izračunaj prostornino piramide $ABCD$.

Mešani produkt vektorjev $\vec{a}$, $\vec{b}$ in $\vec{c}$:

$$(\vec{a}, \vec{b}, \vec{c}) = \underbrace{\vec{a} \cdot (\vec{b} \times \vec{c})}_{\text{ŠTEVILO}}$$

Absolutna vrednost mešanega produkta vektorjev $\vec{a}$, $\vec{b}$ in $\vec{c}$ je prostornina paralelepipeda, napetega na vektorje $\vec{a}$, $\vec{b}$ in $\vec{c}$.

a) $\vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \vec{a}$

$\vec{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \vec{b}$

$\vec{AD} = \vec{r}_D - \vec{r}_A = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \vec{c}$

$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ -10 \end{bmatrix} =$

$\vec{b} \times \vec{c} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 - (-2) \cdot 0 \\ -(2 \cdot 2 - 3 \cdot 0) \\ 2(-2) - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -10 \end{bmatrix}$

$= \; 0 \cdot 4 + 3(-4) + (-3)(-10) = 18$

V paralelepipeda $= |(\vec{a}, \vec{b}, \vec{c})| = |18| = \underline{\underline{18}}$

b) V piramide $= \frac{1}{6} V$ paralelepipeda

$= \frac{1}{6} \cdot 18 = \underline{\underline{3}}$