Matematika VSP, vaje, 06. 11. 2020 (vrste, funkcije)

NALOGA 22.
Naj bo (a_n) zaporedje $a_n = \frac{1}{n(n+1)}$.

a. Poišči $\lim_{n\to\infty} a_n$.
b. S formulo izrazi N-to delno vsoto vrste
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$
tj. $S_N = a_1 + a_2 + \cdots + a_N$. (*Namig:* Zapiši $\frac{1}{n(n+1)}$ kot vsoto parcialnih ulomkov.)
c. Seštej zgornjo vrsto; izračunaj limito delnih vsot $\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Vrsta je "formalna" vsota $a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n$, kjer je (a_n) zaporedje.

Zaporedju (a_n) priredimo zaporedje delnih vsot:

$$S_0 = a_0$$
$$S_1 = a_0 + a_1 = S_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2 = S_1 + a_2$$
$$\vdots$$
$$S_N = a_0 + a_1 + \ldots + a_N = S_{N-1} + a_N.$$

Dobimo (rekurzivno opisano) zaporedje S_N. Vsota vrste je
$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N \quad (\text{če obstaja}).$$

(a) $a_n = \frac{1}{n(n+1)}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{1\cdot\frac{1}{n^2}}{(n^2+n)\cdot\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{1+\frac{1}{n}} = \frac{0}{1+0} = 0.$

$\frac{1}{n^2} \to 0$, $\frac{1}{n} \to 0$

(S predavanj: Če obstaja $\lim_{N\to\infty} S_N$, potem je $\lim_{n\to\infty} a_n = 0$.)

(b) Kaj je $S_N = a_1 + a_2 + \ldots + a_N$, če je $a_n = \frac{1}{n(n+1)}$?

$$S_N = \underbrace{\frac{1}{1\cdot 2}}_{a_1} + \underbrace{\frac{1}{2\cdot 3}}_{a_2} + \underbrace{\frac{1}{3\cdot 4}}_{a_3} + \ldots + \frac{1}{N(N+1)}$$

$$a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{An + A + Bn}{n(n+1)},$$ torej $0\cdot n + 1 = (A+B)n + A$

$A + B = 0$
$A = 1 \ldots B = -1$

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

$$S_N = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}_{a_3} + \cancel{\ldots} + \underbrace{\left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)}_{a_N} = 1 - \frac{1}{N+1}.$$

(c) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \underbrace{\frac{1}{N+1}}_{\to 0}\right) = 1 - 0 = 1.$

OR

NALOGA 23.
Izračunaj vsote naslednjih geometrijskih vrst:

a. $\sum_{n=0}^{\infty} \frac{1}{4^n}$

b. $\sum_{n=1}^{\infty} \frac{10}{3^n}$

c. $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}}$

d. $\frac{3}{2} + 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$

e. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}}$

f. $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^{3n}$, za tiste $x \in \mathbb{R}$, za katere vrsta konvergira.

Geometrijska vrsta je vsota $\sum_{n=0}^{\infty} q^n$, konvergira, če je $|q| < 1$, sicer divergira.

V primeru $|q| < 1$ je $\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \dots = \frac{1}{1-q}$.

(a) $\sum_{n=0}^{\infty} \frac{1^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \underset{q=\frac{1}{4}}{=} \frac{1}{1-1/4} = \frac{1}{3/4} = \frac{4}{3}.$

(b) $\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10}{3} + \frac{10}{3^2} + \frac{10}{3^3} + \dots = 10 \cdot \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right) = 10 \sum_{n=1}^{\infty} \frac{1}{3^n} =$

$= 10 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 10 \cdot \left(\underbrace{\frac{1}{1 - 1/3}}_{= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n} - 1\right) = 10 \cdot \left(\underbrace{\left(\frac{1}{2/3}\right) - 1}_{\frac{1}{2}}\right) = 5.$

$\frac{1}{2/3} = 3/2$

(c) $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}} = \sum_{n=2}^{\infty} \frac{2^n}{\frac{1}{3} \cdot 9^n} = \sum_{n=2}^{\infty} 3 \cdot \left(\frac{2}{9}\right)^n = 3 \sum_{n=2}^{\infty} \left(\frac{2}{9}\right)^n = 3\left(\frac{1}{1-\frac{2}{9}} - 1 - \frac{2}{9}\right)$

$3^{2n-1} = 3^{2n} \cdot 3^{-1} = \frac{1}{3}(3^2)^n$

$= 3 \cdot \frac{(2/9)^2}{1 - 2/9} = \dots$

Koliko je vsota $\sum_{n=n_0}^{\infty} q^n$? $\sum_{n=n_0}^{\infty} q^n = q^{n_0} + q^{n_0+1} + q^{n_0+2} + \dots =$

$= q^{n_0} \underbrace{(1 + q + q^2 + \dots)}_{\frac{1}{1-q}} = \frac{q^{n_0}}{1-q}$

e. $\sum_{n=1}^{\infty}\frac{(-2)^n}{3\cdot 2^{3n-2}}$ $= \frac{4}{3}\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^n = \frac{4}{3}\cdot\frac{(-1/4)^1}{1-(-1/4)} = \frac{4}{3}\cdot\frac{-1/\cancel{4}}{5/\cancel{4}} = \frac{4}{3}\cdot\left(-\frac{1}{5}\right) = -\frac{4}{15}.$

$$\frac{(-2)^n}{3\cdot 2^{3n-2}} = \frac{1}{3}\cdot\frac{(-2)^n}{2^{-2}\cdot 2^{3n}} = \frac{4}{3}\cdot\frac{(-2)^n}{(2^3)^n} = \frac{4}{3}\cdot\left(\frac{-2}{2^3}\right)^n = \frac{4}{3}\left(-\frac{1}{4}\right)^n$$

NALOGA 24. OR

Kateri racionalni ulomek ima decimalni zapis $0.\overline{12} = 0.121212\ldots$? Pomagaj si s primerno geometrijsko vrsto.

$\frac{1}{3} = 0.333\ldots = 0.\overline{3}$

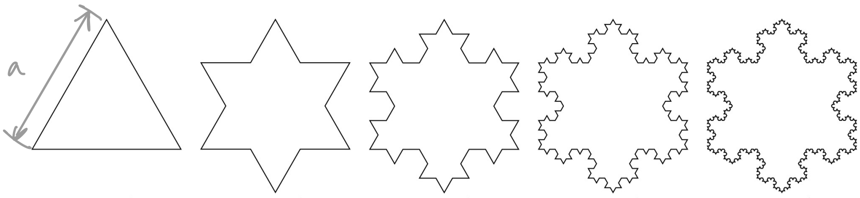
$$0.12121212\ldots = 12\cdot\frac{1}{100} + 12\cdot\frac{1}{100^2} + 12\cdot\frac{1}{100^3} + 12\cdot\frac{1}{100^4} + \ldots$$

$$= 12\cdot\left(\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\ldots\right) = 12\cdot\sum_{n=1}^{\infty}\frac{1}{100^n} = \left(\frac{1}{100}\right)^n$$

$$= 12\cdot\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = 12\cdot\frac{\frac{1}{100}}{\underbrace{1-\frac{1}{100}}_{=\frac{99}{100}}} = 12\cdot\frac{1}{99} = \frac{12}{99} = \frac{4}{33}.$$

NALOGA 25. OR

Kochova snežinka je fraktal, ki ga dobimo z zaporedjem iteracij kot na spodnji sliki.



$n=0$ $n=1$ $n=2$ $n=3$ $n=4$

Recimo, da začnemo z enakostraničnim trikotnikom (pri $n=0$) s stranico dolžine a. Poišči geometrijski vrsti, ki določata ploščino in obseg Kochove snežinke. Seštej ti dve vrsti! Kolikšni sta ploščina in obseg izraženi z a?

Ploščina:

$$\frac{a^2\sqrt{3}}{4} + \frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4} + \frac{4}{9}\cdot\frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4} + \frac{4}{9}\cdot\frac{4}{9}\cdot\frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4} + \ldots =$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4}\underbrace{\left(1+\frac{4}{9}+\left(\frac{4}{9}\right)^2+\ldots\right)}_{=\frac{1}{1-4/9}=\frac{1}{5/9}=\frac{9}{5}} =$$

$$= \frac{a^2\sqrt{3}}{4}\left(1+\overbrace{\frac{1}{\cancel{3}}\cdot\frac{\cancel{9}\,3}{5}}^{\frac{8}{5}}\right) = \frac{2a^2\sqrt{3}}{5}.$$

Obseg: Vsaka iteracija ima 4x toliko stranic kot prejšnja ($\overset{n=0}{3}, \overset{n=1}{12}, \overset{n=2}{36}, \overset{\ldots}{\ldots}$), pri vsaki iteraciji je dolžina stranice $\frac{1}{3}$ dolžine stranic prejšnje iteracije. Torej:

$o_n = 3a\left(\frac{4}{3}\right)^n$, vendar $\lim_{n\to\infty} o_n = \lim_{n\to\infty} 3a\left(\frac{4}{3}\right)^n = \infty$, tj. ne obstaja, obseg je neomejen.