Vaje MAT VSP, 22.10.2020

KOMPLEKSNA ŠTEVILA

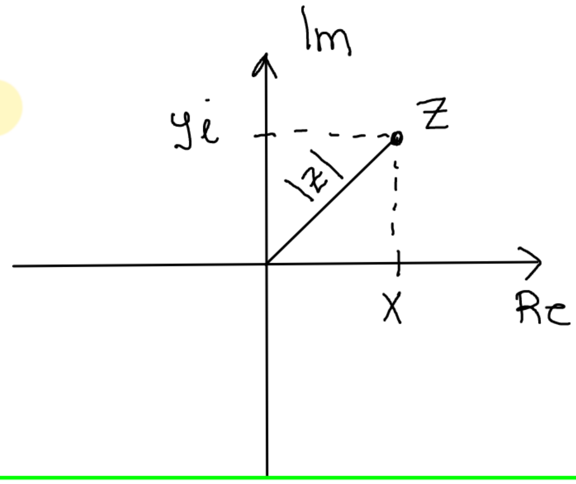
NALOGA 8.
Kompleksno število $z = \frac{1+5i}{1-i}$ zapiši v obliki $z = x + iy$ in izračunaj $|z|$.

$z = x + iy$; $\quad x, y \in \mathbb{R}$

$x = \mathrm{Re}(z)$
$y = \mathrm{Im}(z)$
i … imaginarna enota $\quad i^2 = -1$

$z \cdot \bar{z} = x^2 + y^2$
$\bar{z} = x - iy$

$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$



$$z = \frac{1+5i}{1-i} = \frac{(1+5i)(1+i)}{(1-i)(1+i)} = \frac{1 + i + 5i + \overset{5\cdot(-1)}{5i^2}}{1^2 + (-1)^2} =$$

$$= \frac{-4 + 6i}{2} = \frac{\cancel{2}(-2+3i)}{\cancel{2}} = -2 + 3i$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$

NALOGA 9.
Poišči vse kompleksne rešitve spodnjih (ne)enačb, tj. opiši ali skiciraj množico rešitev v $\mathbb{C}$.

a. $2\bar{z} - z^2 = 0$,
b. $\operatorname{Im}\left(\frac{1}{z}\right) = 1$,
c. $\operatorname{Re} z + \operatorname{Im} z^2 = 2$,
d. $2z^2 - 3\bar{z}^2 = 10i$,
e. $z^2 + (3-1)z = 2i - 2$,
f. $\bar{z} - iz^2 = 0$,
g. $|z - 3 + 2i| = 4$,
h. $|z + i| < |z - 1|$,
i. $|z - 1| + |z + 1| = 4$.

a) $2\bar{z} - z^2 = 0$ $\quad z = x + iy$

$$2(x - iy) - (x + iy)^2 = 0 \qquad i^2 \cdot y^2 = (-1)\cdot y^2 = -y^2$$

$$2x - 2iy - (x^2 + 2xiy + (iy)^2) = 0$$

$$2x - 2iy - x^2 - 2xiy + y^2 = 0$$

$$(2x - x^2 + y^2) + i(-2y - 2xy) = 0$$

$$2x - x^2 + y^2 = 0$$

$$-2y - 2xy = 0$$

$$\downarrow$$

$$-\underbrace{2y}\underbrace{(1 + x)} = 0$$

1. $y = 0$

$$2x - x^2 = 0$$

$$x(2 - x) = 0$$

$$x = 0 \qquad 2 - x = 0$$

$$x = 2$$

$$\boxed{\begin{aligned} z_1 &= 0 + 0i = 0 \\ z_2 &= 2 + 0i = 2 \end{aligned}}$$

2. $1 + x = 0$

$$x = -1$$

$$2(-1) - (-1)^2 + y^2 = 0$$

$$-2 - 1 + y^2 = 0$$

$$-3 + y^2 = 0$$

$$y^2 = 3 \quad / \sqrt{}$$

$$y = \pm\sqrt{3}$$

$$\boxed{\begin{aligned} z_3 &= -1 + \sqrt{3}i \\ z_4 &= -1 - \sqrt{3}i \end{aligned}}$$

b) $\operatorname{Im}\left(\frac{1}{z}\right)=1$ $\boxed{z=x+iy}$

$$\operatorname{Im}\left(\frac{1}{x+iy}\right)=1$$

$$\operatorname{Im}\left(\frac{1\,(x-iy)}{(x+iy)(x-iy)}\right)=1$$

$$\operatorname{Im}\left(\frac{x-iy}{x^2+y^2}\right)=1$$

$$\operatorname{Im}\left(\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}\right)=1$$

$$-\frac{y}{x^2+y^2}=1 \;/\;(x^2+y^2)$$

$$-y=x^2+y^2$$

$$x^2+\underbrace{y^2+1y}=0$$

Enačba krožnice:

$$(x-a)^2+(y-b)^2=r^2$$

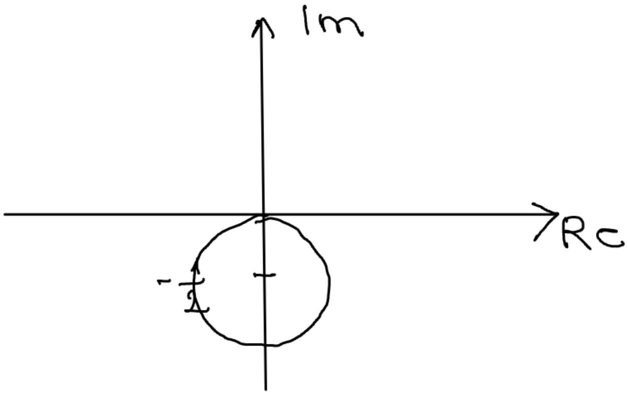
$S(a,b)$... središče krožnice

r... polmer

$$x^2+\underbrace{\left(y+\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2}=0$$

$$x^2+\underbrace{\left(y+\tfrac{1}{2}\right)^2}_{y-\left(-\frac{1}{2}\right)}=\left(\tfrac{1}{2}\right)^2$$

$$S\left(0,-\tfrac{1}{2}\right) \qquad r=\frac{1}{2}$$



Rešitve so vsa kompleksna števila, ki ležijo na krožnici.

c) $\mathrm{Re}\, z + \mathrm{Im}\, z^2 = 2$ $\qquad \boxed{z = x + iy}$

$$x + \mathrm{Im}\,(x+iy)^2 = 2 \qquad i^2 \cdot y^2 = -1 \cdot y^2 = -y^2$$

$$x + \mathrm{Im}\,(x^2 + 2xiy + (iy)^2) = 2$$

$$x + 2xy = 2$$

$$2xy = 2 - x \quad /:2x$$

$$y = \frac{2-x}{2x}$$

g) $|z - 3 + 2i| = 4$ $\qquad \boxed{z = x + iy}$

$$|x + iy - 3 + 2i| = 4$$

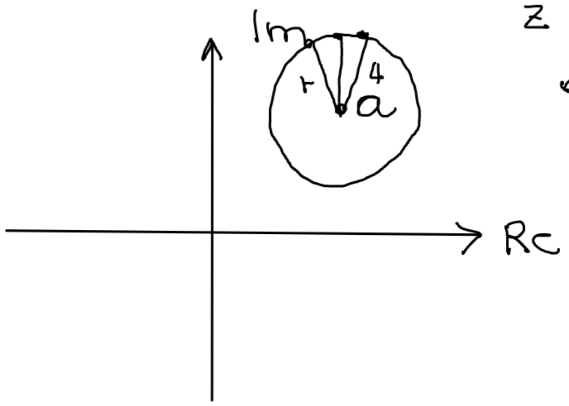
$$|\underbrace{(x-3)}_{\mathrm{Re}} + i\underbrace{(y+2)}_{\mathrm{Im}}| = 4$$

$$\sqrt{(x-3)^2 + (y+2)^2} = 4 \quad /^2$$

$$(x-3)^2 + (y+2)^2 = 4^2$$

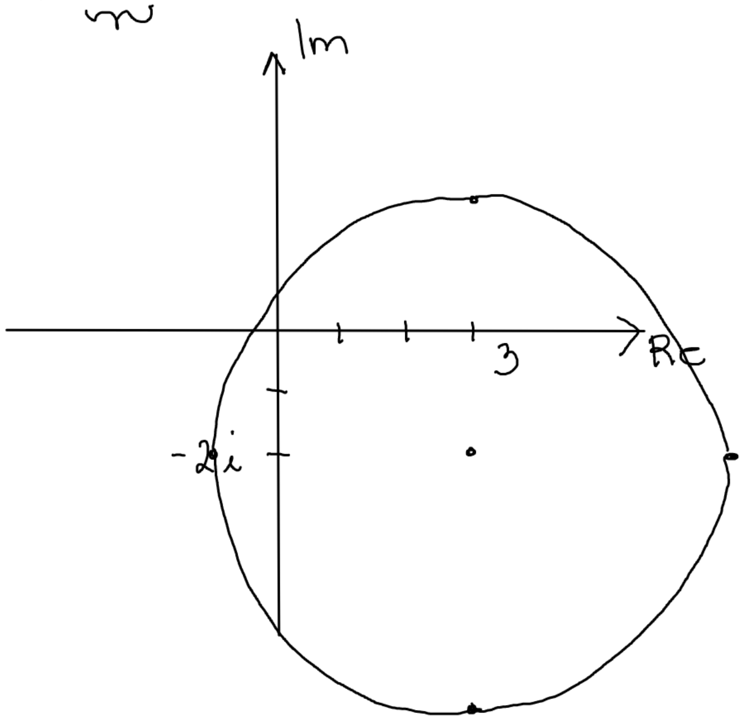
$$S(3, -2) \qquad r = 4$$

$|z-a| = \overset{r}{4}$ $\longrightarrow$ vsa kompleksna števila z, ki so od a oddaljena za 4 enote



$|z-3+2i| = 4$

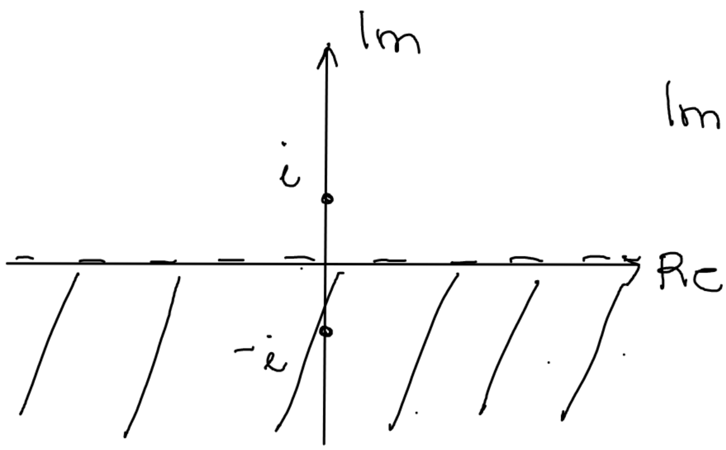
$|z-(3-2i)| = 4$



h) $|z+i| < |z-i|$

razdalja od z do $(-i)$

razdalja od z do i



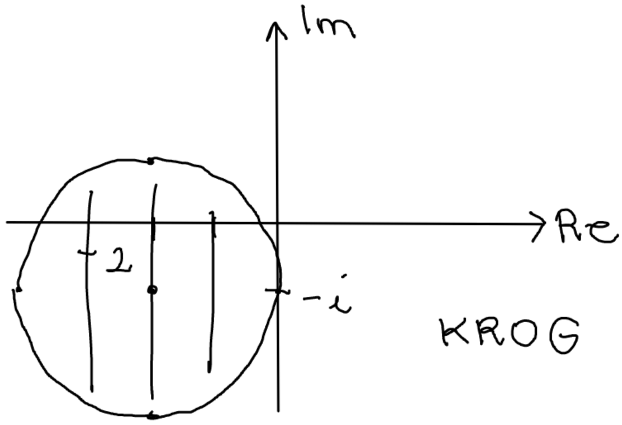
$\mathrm{Im}(z) < 0$

NALOGA 10.
V kompleksni ravnini skiciraj množice rešitev spodnjih neenačb:

a. $|\overline{z}+2-i| \leq 2$,
b. $\mathrm{Re}(\overline{z}+2-i) \leq 2$,
c. $\mathrm{Im}(\overline{z}+2-i) \leq 2$.

a) $|\overline{z}+2-i| \leq 2$ $\boxed{z = x+iy}$

$$|x-iy+2-i| \leq 2$$
$$|(x+2)-i(y+1)| \leq 2$$
$$\sqrt{(x+2)^2+(y+1)^2} \leq 2 \quad /^2$$
$$(x+2)^2+(y+1)^2 \leq 2^2$$
$$S(-2,-1) \qquad r = 2$$



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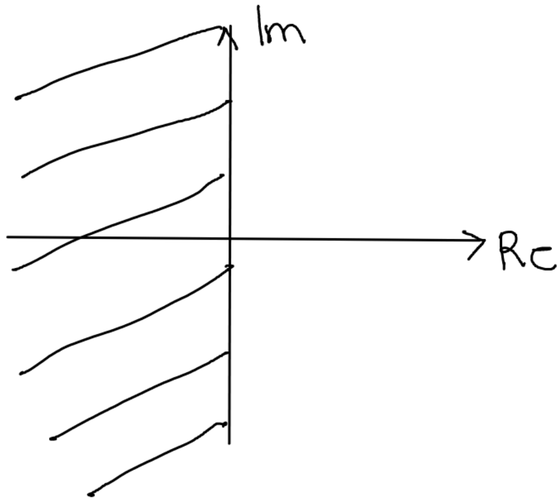
b) $\mathrm{Re}(\overline{z}+2-i) \leq 2$ $\boxed{z = x+iy}$

$$\mathrm{Re}(x-iy+2-i) \leq 2$$
$$x+2 \leq 2$$
$$x \leq 0$$

POLRAVNINA

$$\mathrm{Re}(z) \leq 0$$



NALOGA 13.

Prevedi v polarno obliko, nato pa z uporabo Eulerjeve formule izračunaj

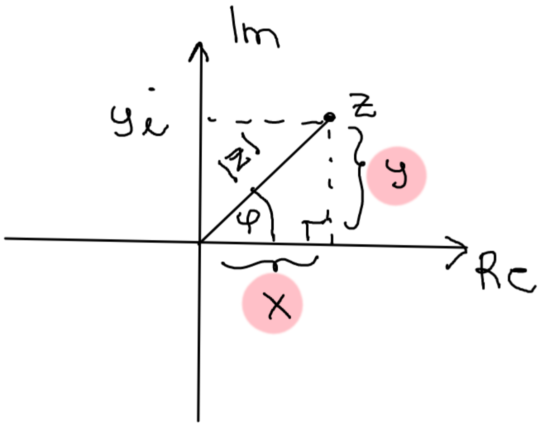
a. $\left(-\frac{1}{2}+\frac{i}{2}\right)^8$,

b. $\left(1+i\sqrt{3}\right)^{20}$,

c. $(1-i)^{20}$,

d. $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$.

POLARNI ZAPIS: $z = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$



$z = x + iy$

$\varphi = \mathrm{Arg}(z)$

je določen do mnogokratnika celega kota 2π natančno

$\tan\varphi = \frac{y}{x}$

$z^n = |z|^n \cdot e^{in\varphi}$

a) $\left(-\frac{1}{2}+\frac{i}{2}\right)^8$

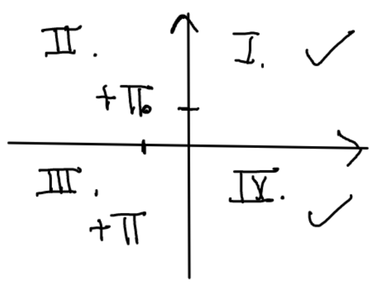
$z = -\frac{1}{2} + \frac{1}{2}i$

$|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan\varphi = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$

$\varphi = \arctan(-1) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

arc tan:



$$z = \frac{\sqrt{2}}{2} \cdot e^{i \frac{3\pi}{4}}$$

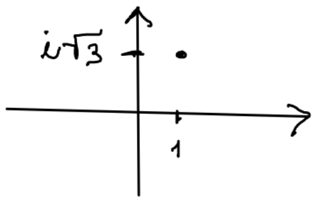
$$6\pi = 3 \cdot 2\pi = 0$$

$$z^8 = |z|^8 \cdot e^{i \cdot 8 \cdot \varphi}$$

$$z^8 = \left(\frac{\sqrt{2}}{2}\right)^{\overset{= 4\cdot 2}{8}} e^{i \cdot 8 \cdot \frac{3\pi}{4}} = \left(\frac{2}{4}\right)^4 \cdot e^{i 6\pi} = \left(\frac{1}{2}\right)^4 \cdot e^{i \cdot 0}$$

$$= \frac{1}{16} \cdot 1 = \frac{1}{16}$$

b) $(1+i\sqrt{3})^{20}$



$$z = 1 + i\sqrt{3}$$

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\varphi = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\varphi = \operatorname{arc\,tg}(\sqrt{3}) = \frac{\pi}{3}$$ ✓ 1. kvadrant

$$z = 1 + i\sqrt{3} = 2 \cdot e^{i \frac{\pi}{3}}$$

$$z^{20} = (1+i\sqrt{3})^{20} = |z|^{20} \cdot e^{i \cdot 20 \cdot \varphi}$$

$$= 2^{20} \cdot e^{i \cdot 20 \cdot \frac{\pi}{3}} = 2^{20} \cdot e^{i \frac{2\pi}{3}}$$

$$\frac{20\pi}{3} = \frac{18\pi + 2\pi}{3} = 3 \cdot 2\pi + \frac{2\pi}{3}$$

NALOGA 16.
Poišči naslednja števila:

a. $\sqrt{1+i}$,
b. $\sqrt[3]{-27+27i}$,
c. $\sqrt[5]{-32i}$,
d. $\sqrt[3]{-1+i\sqrt{3}}$.

a) $z = \sqrt{1+i} \quad /^2$

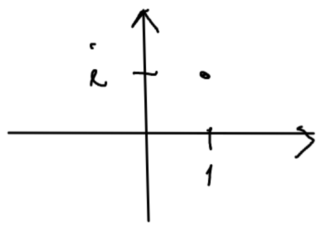
$z^2 = 1+i$

Rešitve enačbe $z^m = a$
- a zapišemo v polarni obliki
- dobimo m različnih rešitev:

$$z_k = \sqrt[m]{|a|}\, e^{i \frac{\varphi + 2k\pi}{m}} \qquad k = 0, 1, \dots, m-1$$

(rešitve ležijo na ogliščih pravilnega m-kotnika)

$a = 1+i$



$|a| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\tan\varphi = \frac{1}{1} \begin{matrix} \text{Im}(a) \\ \text{Re}(a) \end{matrix} = 1 \implies \varphi = \arctan(1) = \frac{\pi}{4} \checkmark$$

1. kvadrant

$a = 1+i = \sqrt{2}\, e^{i \cdot \frac{\pi}{4}}$

$$z_k = \sqrt[2]{\sqrt[2]{2}} \cdot e^{i \frac{\frac{\pi}{4} + 2k\pi}{2}} \qquad k = 0, 1$$

$$z_0 = \sqrt[4]{2}\, e^{i \frac{\frac{\pi}{4} + 2 \cdot 0 \cdot \pi}{2}} = \sqrt[4]{2}\, e^{i \frac{\pi}{8}}$$

$$\left[\frac{\frac{\pi}{4}}{\frac{2}{1}}\right] = \frac{\pi \cdot 1}{4 \cdot 2} = \frac{\pi}{8}$$

$$Z_1 = \sqrt[4]{2}\; e^{i\frac{\frac{\pi}{4}+2\cdot 1\cdot \pi}{2}} = \sqrt[4]{2}\; e^{i\frac{\frac{\pi}{4}+\frac{8\pi}{4}}{2}} =$$

$$= \sqrt[4]{2}\; e^{i\frac{\frac{9\pi}{4}}{\frac{2}{1}}} = \sqrt[4]{2}\; e^{i\frac{9\pi}{8}}$$

c) $z = \sqrt[5]{-32i} \;/^{5}$

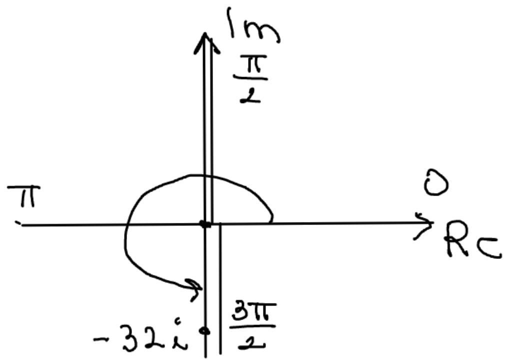
$z^5 = -32i$

$a = -32i = 0 - 32i$

$|a| = \sqrt{0^2 + (-32)^2} = \sqrt{(-32)^2} = 32$

$\tan\varphi = \frac{-32}{0}$ //

$\varphi = \frac{3\pi}{2}$



$a = 32 \cdot e^{i\frac{3\pi}{2}}$

$$Z_k = \sqrt[5]{32}\; e^{i\frac{\frac{3\pi}{2}+2k\pi}{5}} \qquad k = 0, 1, 2, 3, 4$$

$$Z_0 = 2\; e^{i\frac{\frac{3\pi}{2}+2\cdot 0\cdot \pi}{5}} = 2\cdot e^{i\frac{\frac{3\pi}{2}}{\frac{5}{1}}} = 2\cdot e^{i\frac{3\pi}{10}}$$

$$Z_1 = 2\cdot e^{i\frac{\frac{3\pi}{2}+2\cdot\pi}{5}} = 2\cdot e^{i\frac{\frac{3\pi+4\pi}{2}}{5}} =$$

$$= 2\cdot e^{i\frac{7\pi}{10}}$$

$Z_2 = 2e^{i\frac{11\pi}{10}}$ $\qquad$ $Z_4 = 2\cdot e^{i\frac{19\pi}{10}}$

$Z_3 = 2\cdot e^{i\frac{15\pi}{10}}$