Matematika VSP, vaje, 23. 10. 2020 (Kompleksna števila)

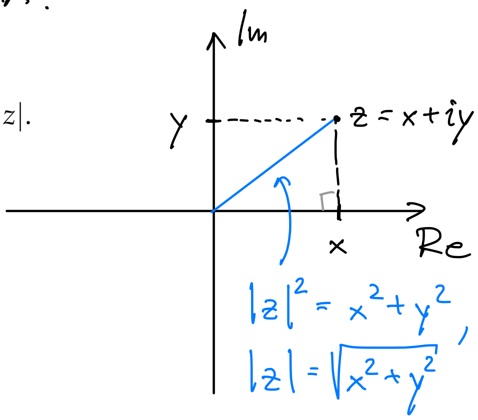
Kaj je rešitev enačbe $z^2+1=0$?

$z^2=-1 \ \ldots \ \text{"} z = \pm\sqrt{-1} \text{"} \ \ldots \ z=\pm i$

$\uparrow$

imaginarna enota, ima lastnost $i^2=-1$.

Kompleksna števila so oblike $x+iy$; $x,y\in\mathbb{R}$.



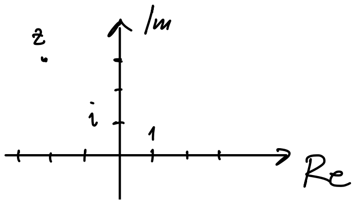
NALOGA 8.
Kompleksno število $z=\frac{1+5i}{1-i}$ zapiši v obliki $z=x+iy$ in izračunaj $|z|$.

$$z=\frac{1+5i}{1-i}=\frac{(1+5i)}{(1-i)}\cdot\frac{(1+i)}{(1+i)}=\frac{1+5i+i+5i^2}{1-i+i-i^2}=$$

$1+i=\overline{1-i}$

$$=\frac{1+6i-5}{1-(-1)}=\frac{-4+6i}{2}=-2+3i$$

$$|z|=|-2+3i|=\sqrt{(-2)^2+3^2}=\sqrt{4+9}=\sqrt{13}.$$



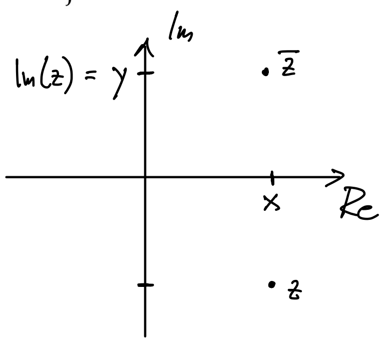
NALOGA 9.
Poišči vse kompleksne rešitve spodnjih (ne)enačb, tj. opiši ali skiciraj množico rešitev v $\mathbb{C}$.

a. $2\bar{z}-z^2=0$,
b. $\operatorname{Im}\left(\frac{1}{z}\right)=1$,
c. $\operatorname{Re} z+\operatorname{Im} z^2=2$,
d. $2z^2-3\bar{z}^2=10i$,
e. $z^2+(3-1)z=2i-2$,
f. $\bar{z}-iz^2=0$,
g. $|z-3+2i|=4$,
h. $|z+i|<|z-1|$,
i. $|z-1|+|z+1|=4$.

$\overline{x+iy}=x-iy$

$\operatorname{Re}(x+iy)=x$

$\operatorname{Im}(x+iy)=y$



(a) $2\bar{z}-z^2=0$, $z=x+iy$, $x,y\in\mathbb{R}$

$$2(x-iy)-(x+iy)^2=0 \ \ldots \ 2(x-iy)-(x^2+2ixy+\overbrace{(iy)^2}^{i^2\cdot y^2=-y^2})=0 \ \ldots$$

$$\ldots \ 2x-2iy-x^2-2ixy+y^2=0 \ \ldots \ \underbrace{(2x-x^2+y^2)}_{=0}+i\underbrace{(-2y-2xy)}_{=0}=0$$

Dobimo: $2x-x^2+y^2=0$

$-2y-2xy=0 \ \ldots \ -2y(1+x)=0$, tj. $y=0$ ali $1+x=0$

oz. $x=-1$.

Torej $y = 0$ ali $x = -1$. To vstavimo v prvo enačbo.

$$2x - x^2 = 0$$
$$x(2-x) = 0,$$
tj. $x_1 = 0$
ali $x_2 = 2$

$$2\cdot(-1) - (-1)^2 + y^2 = 0$$
$$-3 + y^2 = 0 \quad \text{oz.} \quad y^2 = 3 \;\ldots\; y_{1,2} = \pm\sqrt{3}.$$

$z_1 = 0 = 0 + 0i$
$z_2 = 2 = 2 + 0i$

$z_3 = -1 + i\sqrt{3}$ $\quad (\bar{z}_3 = -1 - i\sqrt{3})$
$z_4 = -1 - i\sqrt{3}$

(b) $\operatorname{Im}\left(\frac{1}{z}\right) = 1 \quad \ldots \quad z = x + iy$

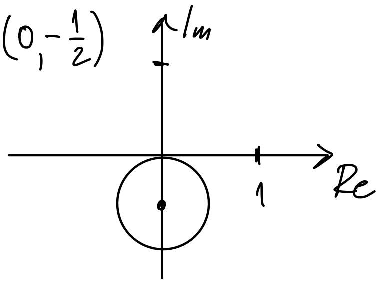
$$\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} \;\ldots\; \operatorname{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2} = 1 \;\Big/ \cdot (x^2+y^2)$$

$$-y = x^2 + y^2 \;\ldots\; x^2 + \underbrace{y^2 + y}_{y^2+y+\frac{1}{4}-\frac{1}{4}} = 0 \;\ldots\; x^2 + \underbrace{y^2 + 2\cdot y\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2}_{\left(y+\frac{1}{2}\right)^2} = \frac{1}{4}$$

$$x^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$(x-p)^2 + (y-q)^2 = r^2 \leftarrow$ krožnica s središčem v (p,q) in polmerom r

krožnica s središčem v $\left(0, -\frac{1}{2}\right)$ in polmeram $\frac{1}{2}$



(h) $|z+i|^2 < |z-1|^2 \qquad z = x + iy$

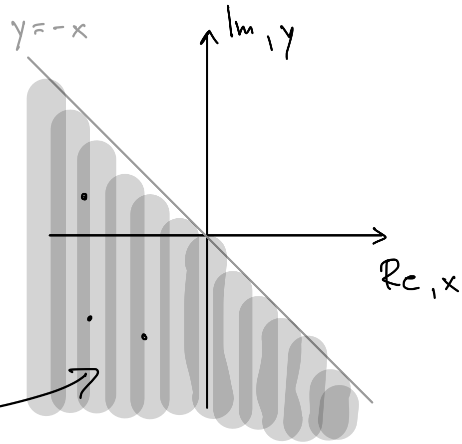
$$|z+i| = |x+iy+i| = |x + i(y+1)| = \sqrt{x^2 + (y+1)^2}$$
$$|z-1| = |(x-1) + iy| = \sqrt{(x-1)^2 + y^2}$$

Torej rešujemo $x^2 + (y+1)^2 < (x-1)^2 + y^2$

$$\cancel{x^2} + \cancel{y^2} + 2y + \cancel{1} < \cancel{x^2} - 2x + \cancel{1} + \cancel{y^2}$$

$$2y < -2x \quad \text{oz.} \quad y < -x$$

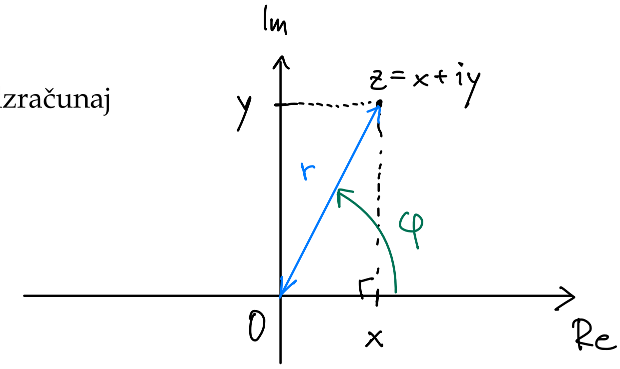
Vsa števila v tem delu rešijo neenačbo.



NALOGA 13.

Prevedi v polarno obliko, nato pa z uporabo Eulerjeve formule izračunaj

a. $\left(-\frac{1}{2}+\frac{i}{2}\right)^8$,
b. $\left(1+i\sqrt{3}\right)^{20}$,
c. $(1-i)^{20}$,
d. $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$.

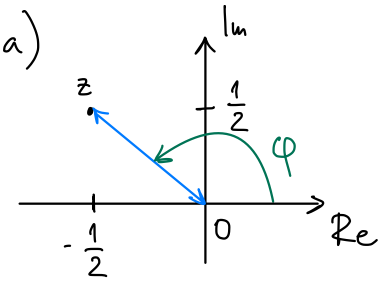


$$z = x+iy = r\cos\varphi + i\cdot r\sin\varphi = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\tan\varphi = \frac{y}{x}$$

(a)



$$\varphi = \frac{3\pi}{4} = 135^\circ$$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Eulerjeva formula:

$$\left(r(\cos\varphi + i\sin\varphi)\right)^n = r^n\left(\cos(n\varphi) + i\sin(n\varphi)\right) = r^n e^{in\varphi}$$

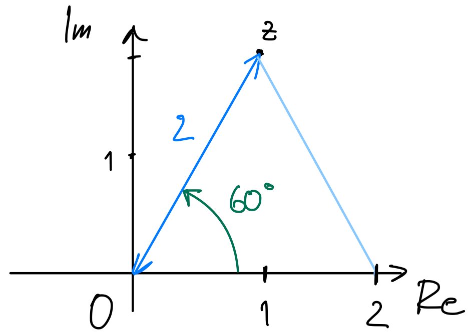
$$\left(-\frac{1}{2}+\frac{i}{2}\right)^8 = \left(\frac{\sqrt{2}}{2}\right)^8\left(\cos\left(8\cdot\frac{3\pi}{4}\right) + i\sin\left(8\cdot\frac{3\pi}{4}\right)\right) = \quad \Big| \; -\frac{1}{2}+\frac{i}{2} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{2^4}{2^8}\cdot\left(\underbrace{\cos(6\pi)}_{1} + i\underbrace{\sin(6\pi)}_{0}\right) = \frac{1}{2^4}\cdot 1 = \frac{1}{16}.$$

(b) $\left(1+i\sqrt{3}\right)^{20}$, $z = 1+i\sqrt{3}$, $z^{20} = ?$

$$r = |z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\tan\varphi = \frac{\sqrt{3}}{1} = \sqrt{3}, \quad \varphi = 60^\circ = \frac{\pi}{3}$$



$$z = 2e^{i\pi/3} \quad \ldots \quad z^{20} = 2^{20}e^{i\frac{20\pi}{3}} =$$

$$\frac{20\pi}{3} = 6\pi + \frac{2\pi}{3}$$

$$= 2^{20}\left(\cos\left(\frac{20\pi}{3}\right) + i\sin\left(\frac{20\pi}{3}\right)\right) =$$

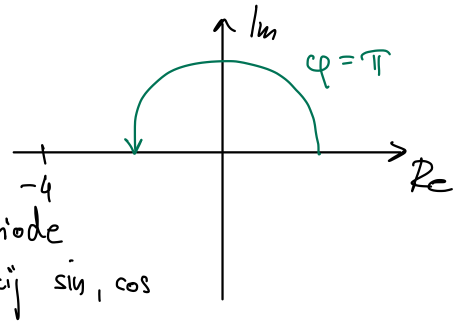
$$= 2^{20}\cdot\left(\underbrace{\cos\left(\frac{2\pi}{3}\right)}_{=-\frac{1}{2}} + i\underbrace{\sin\left(\frac{2\pi}{3}\right)}_{=\frac{\sqrt{3}}{2}}\right) = 2^{20}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) =$$

$$= \underline{2^{19}\left(-1+i\sqrt{3}\right)}.$$

NALOGA 15.

$\mathcal{OR}$

Reši enačbo $z^4+4=0$, nato pa razstavi polinom z^4+4 na dva kvadratna faktorja z realnimi koeficienti.



$$z^4+4=0 \quad \ldots \quad z^4=-4=4e^{i\pi}$$

$$z=re^{i\varphi} \longrightarrow (re^{i\varphi})^4 = 4e^{i\pi}$$

$$r^4e^{4i\varphi}=4e^{i\pi}$$

večkratnik periode kotnih funkcij sin, cos

$$\ldots \quad r^4=4 \quad \text{in} \quad 4\varphi=\pi+\underbrace{2k\pi}$$

$$r=\sqrt[4]{4}=\sqrt{2} \qquad \varphi_k=\frac{\pi}{4}+\frac{k\pi}{2}, \quad k\in\mathbb{Z}$$

$$\varphi_0=\pi/4, \quad \varphi_1=\frac{3\pi}{4}, \quad \varphi_2=\frac{5\pi}{4}, \quad \varphi_3=\frac{7\pi}{4}$$

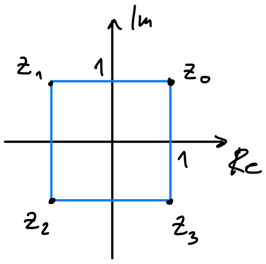
Rešitve $z^4+4=0$ so števila:

$$z_0=\sqrt{2}\,e^{i\pi/4}=\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=1+i$$

$$z_1=\sqrt{2}\,e^{i\frac{3\pi}{4}}=1-i$$

$$z_2=\sqrt{2}\,e^{i\frac{5\pi}{4}}=-1-i$$

$$z_3=\sqrt{2}\,e^{i\frac{7\pi}{4}}=-1+i$$



z^4+4 lahko torej razstavimo kot:

$$z^4+4=(z-z_0)(z-z_1)(z-z_2)(z-z_3)=(z-1-i)(z-1+i)(z+1+i)(z+1-i)=$$

$$=(z^2-2z+2)(z^2+2z+2).$$