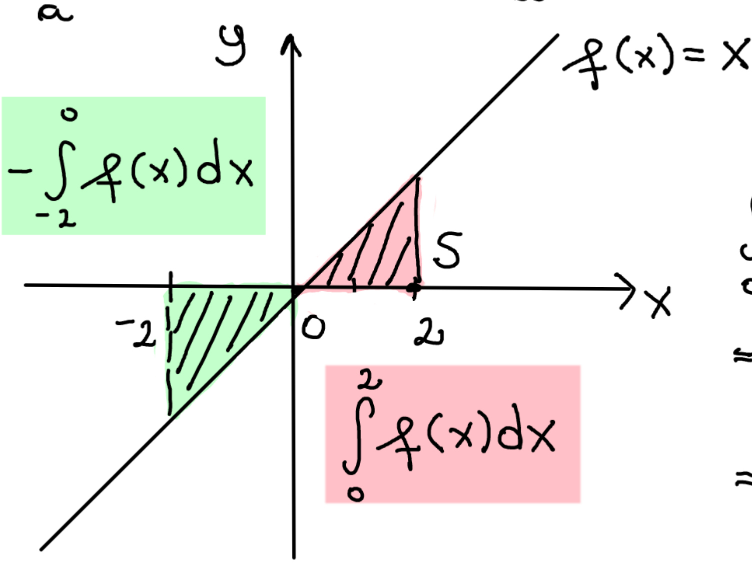
Vaje MAT VSP, 9.12.2020

DOLOČENI INTEGRAL

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Newton-Leibnizova formula



$$\int_0^2 f(x)\,dx = \int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = \underline{\underline{2}} =$$

ploščina območje med grafom funkcije f in x osjo na intervalu $[0, 2]$

$$\int_{-2}^{0} f(x)\,dx = \int_{-2}^{0} x\,dx = \frac{x^2}{2}\Big|_{-2}^{0} = \left(\frac{0^2}{2} - \frac{(-2)^2}{2}\right) = \underline{\underline{-2}} =$$

$-$ ploščina območja med x osjo in grafom funkcije f na intervalu $[-2, 0]$

4. Izračunaj določene integrale

(a) $\int_1^2 2x(x^2+1)^2\,dx$

(b) $\int_0^{\pi/3} \tan(x)\,dx$

(c) $\int_0^{\pi/3} \frac{x}{\cos^2(x)}\,dx$

(d) $\int_1^e \frac{1+\log x}{x}\,dx$,

(e) $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$

(f) $\int_{-\pi}^{\pi} x\sin x\,dx$,

a) 1. način:

$$\int_1^2 2x(x^2+1)^2 dx = \int_1^2 2x(x^4+2x^2+1)\,dx =$$

$$= \int_1^2 (2x^5+4x^3+2x)\,dx =$$

$$= \left(\cancel{2}\cdot\frac{x^6}{\cancel{6}\cdot 3} + \cancel{4}\frac{x^4}{\cancel{4}} + \cancel{2}\frac{x^2}{\cancel{2}}\right)\Bigg|_1^2 =$$

$$= \left(\frac{64}{3}+16+4\right) - \left(\frac{1}{3}+1+1\right) =$$

$$= \frac{63}{3} + 18 = 21+18 = \underline{\underline{39}}$$

2. način (uvedba nove spremenljivke):

$$\int_1^2 2x(\underbrace{x^2+1}_{u})^2 dx = \int_2^5 u^2\,du = \frac{u^3}{3}\Bigg|_2^5 =$$

$$\left[\begin{array}{ll} u = x^2+1 \longrightarrow & S: x=1 \longrightarrow u = 1^2+1 = 2 \\ du = 2x\,dx & Z: x=2 \longrightarrow u = 2^2+1 = 5 \end{array}\right]$$

$$= \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = \underline{\underline{39}}$$

c) $$\int_0^{\frac{\pi}{3}} \frac{x}{\cos^2 x}\,dx = x\tan x \Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx \ominus$$

PER PARTES: $$\left[\begin{array}{lcl} u = x & \xrightarrow{\ '\ } & du = dx \\ dv = \frac{1}{\cos^2 x}\,dx & \xrightarrow{\ \int\ } & v = \tan x \end{array}\right]$$

$$\ominus\ \frac{\pi}{3}\cdot\tan\frac{\pi}{3} - \underbrace{0\cdot\tan 0}_{0} - \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx =$$

NOVA SPREMENLJIVKA $$\left[\begin{array}{ll} u = \cos x & \longrightarrow S: x=0 \to u = \cos 0 = 1 \\ du = -\sin x\,dx & \quad\ Z: x = \frac{\pi}{3} \to u = \cos\frac{\pi}{3} = \frac{1}{2} \\ -du = \sin x\,dx & \end{array}\right]$$

$$= \frac{\pi}{3}\sqrt{3} - \int_1^{\frac{1}{2}} -\frac{du}{u} =$$

$$= \frac{\sqrt{3}\pi}{3} + \int_1^{\frac{1}{2}} \frac{du}{u} =$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$$= \frac{\sqrt{3}\pi}{3} - \int_{\frac{1}{2}}^{1} \frac{du}{u} =$$

$$= \frac{\sqrt{3}\pi}{3} - \log|u| \Big|_{\frac{1}{2}}^{1} =$$

$$= \frac{\sqrt{3}\pi}{3} - (\underbrace{\log 1}_{0} - \log\frac{1}{2}) =$$

$$= \frac{\sqrt{3}\pi}{3} + \log\frac{1}{2} = \frac{\sqrt{3}\pi}{3} + \log 2^{-1} = \frac{\sqrt{3}\pi}{3} - \log 2$$

5. Izračunaj

$$\int_{-1}^{2} f(x)\,dx, \quad \text{kjer je} \quad f(x) = \begin{cases} x^2+1, & \text{če } x \geq 1, \\ -2x+4, & \text{če } x < 1. \end{cases}$$

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx =$$

$$= \int_{-1}^{1} (-2x+4)\,dx + \int_{1}^{2} (x^2+1)\,dx =$$

$$= \left(-\cancel{2}\cdot\frac{x^2}{\cancel{2}} + 4\cdot x\right)\Big|_{-1}^{1} + \left(\frac{x^3}{3} + x\right)\Big|_{1}^{2} =$$

$$= \left(-\cancel{2}\cdot\frac{1}{\cancel{2}} + 4\right) - (-1-4) + \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) =$$

$$= 8 + \frac{14}{3} - \frac{4}{3} = \frac{24+14-4}{3} = \underline{\underline{\frac{34}{3}}}$$

6. Naj bo $f(x) = \int_0^x 3\sin(2t)\,dt$.

(a) Izračunaj $f(2)$.

(b) Določi $f'(\frac{\pi}{8})$.

Zveza med določenim in nedoločenim integralom:

$$F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$$

Sledi: F je nedoločeni integral funkcije f.

$$F(b) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$$

a) $f(2)=\int_0^2 3\sin(\underbrace{2t}_{u})\,dt = \int_0^4 3\cdot\sin(u)\,\frac{du}{2} =$

$$\left[\begin{array}{ll} u=2t & \longrightarrow \text{meje:} \\ du=2\,dt \quad /:2 & S: x=0 \to u=2\cdot 0=0 \\ \frac{du}{2}=dt & Z: x=2 \to u=2\cdot 2=4 \end{array}\right]$$

$= \frac{3}{2}\int_0^4 \sin(u)\,du = \frac{3}{2}\left(-\cos u\right)\Big|_0^4 =$

$= \frac{3}{2}\left(-\cos 4-(-\cos 0)\right) = \frac{3}{2}\left(-\cos 4+1\right)$

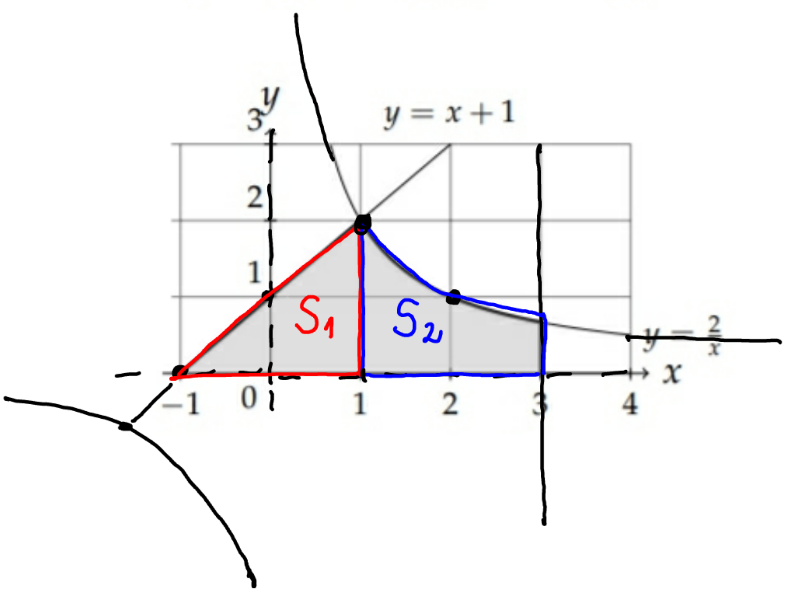
$= \frac{3}{2}\left(1-\cos 4\right)$

b) $f'\left(\frac{\pi}{8}\right)$

$f'(x)=3\sin(2x)$

$f'\left(\frac{\pi}{8}\right)=3\cdot\sin\left(2\cdot\frac{\pi}{8}\right)=3\cdot\sin\left(\frac{\pi}{4}\right)=3\cdot\frac{\sqrt{2}}{2}$

7. Izračunaj ploščino območja, ki ga omejujejo krivulje $y=2/x$, $y=x+1$, $x=3$ in os x.



$y = \frac{2}{x} = 2 \cdot x^{-1}$

pol: $x = 0$

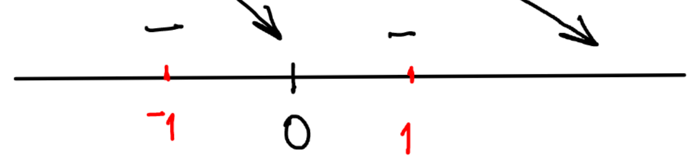
asimptota: $\lim\limits_{x \to \infty} \frac{2}{x} = 0$

ekstremi: $y' = 2(-1)x^{-2} = -\frac{2}{x^2} = 0$

$-2 = 0$ //

$\Rightarrow$ ni ekstremov

naraščanje, padanje: y'



PREČIŠČE $y = x+1$ in $y = \frac{2}{x}$:

$$y = y$$

$$x + 1 = \frac{2}{x} \quad / \cdot x$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x_1 = -2 \qquad x_2 = 1$$

$$S = S_1 + S_2 = \int_{-1}^{1} (x+1)\,dx + \int_{1}^{3} \frac{2}{x}\,dx =$$

$$= \left(\frac{x^2}{2} + x\right)\Bigg|_{-1}^{1} + 2\log|x|\,\Bigg|_{1}^{3} =$$

$$= \left(\cancel{\tfrac{1}{2}} + 1\right) - \left(\cancel{\tfrac{1}{2}} - 1\right) + 2\log 3 - 2\overset{0}{\log 1} =$$

$$= 2 + 2\log 3$$

8. Izračunaj ploščino območja, ki ga omejujeta krivulji $y = 4x - x^2$ in $y = x$.

PRESEČIŠČA:

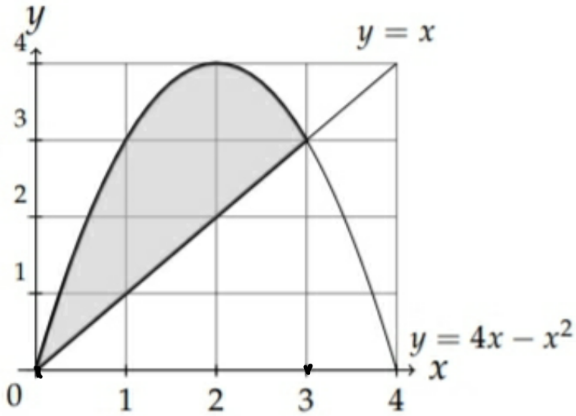
$$y = y$$

$$4x - x^2 = x$$

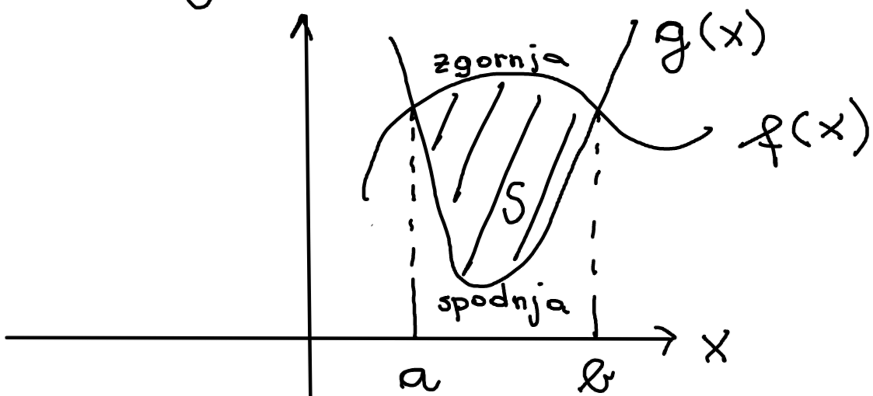
$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x_1 = 0 \qquad x_2 = 3$$



Ploščina območja med krivuljama f in g:

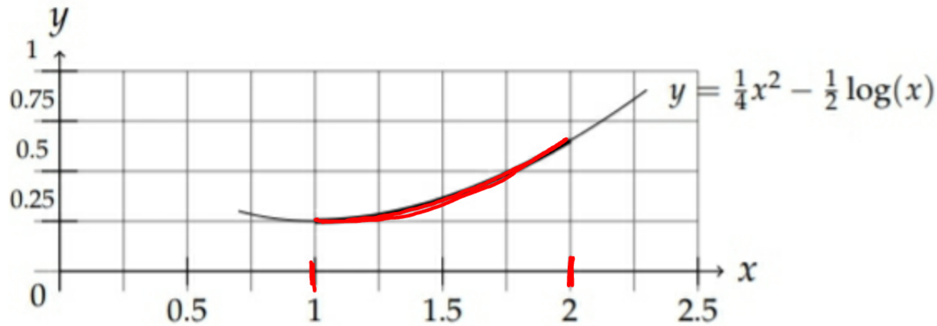


$$S = \int_a^b (\underbrace{f(x)}_{\text{zgornja}} - \underbrace{g(x)}_{\text{spodnja}})\,dx$$

$$S = \int_0^3 (4x - x^2 - x)\,dx = \int_0^3 (-x^2 + 3x)\,dx =$$

$$= \left(-\frac{x^3}{3} + 3 \cdot \frac{x^2}{2}\right)\Big|_0^3 = \left(-9 + \frac{27}{2}\right) = \frac{-18 + 27}{2} = \underline{\underline{\frac{9}{2}}}$$

11. Izračunaj dolžino loka krivulje $y = \frac{1}{4}x^2 - \frac{1}{2}\log(x)$ med točkama, kjer je $x = 1$ in $x = 2$. Uporabi formulo $l = \int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx$.



Dolžina loka krivulje y na intervalu $[x_1, x_2]$ (med točkama $x = x_1$ in $x = x_2$):

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx$$

$$y = \frac{1}{4}x^2 - \frac{1}{2}\log(x)$$

$$y' = \frac{1}{4}\cdot 2x - \frac{1}{2}\cdot\frac{1}{x} = \frac{x}{2} - \frac{1}{2x} = \frac{x^2-1}{2x}$$

$$\ell = \int_1^2 \sqrt{1 + \left(\frac{x^2-1}{2x}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{(x^2-1)^2}{(2x)^2}}\, dx =$$

$$= \int_1^2 \sqrt{\frac{4x^2 + x^4 - 2x^2 + 1}{(2x)^2}}\, dx = \int_1^2 \sqrt{\frac{x^4 + 2x^2 + 1}{(2x)^2}}\, dx =$$

$$= \int_1^2 \sqrt{\frac{(x^2+1)^2}{(2x)^2}}\, dx = \int_1^2 \frac{x^2+1}{2x}\, dx =$$

$$= \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left(\frac{1}{2}\,\frac{x^2}{2} + \frac{1}{2}\log|x|\right)\Bigg|_1^2 =$$

$$= 1 + \frac{1}{2}\log 2 - \left(\frac{1}{4} + \frac{1}{2}\underbrace{\log 1}_{0}\right) =$$

$$= \frac{3}{4} + \frac{1}{2}\log 2$$

13. Izračunaj prostornine vrtenin, ki jih dobiš, če:

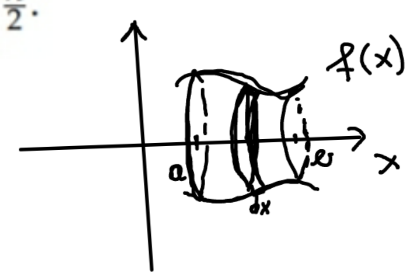
(a) parabolo $y = 1 - x^2$ zavrtiš okrog x-osi med obema ničlama,

(b) graf funkcije $\cos x$ zavrtiš okrog x-osi med $-\frac{\pi}{2}$ in $\frac{\pi}{2}$.

Pomagaj si s formulo

$$\boxed{V_x = \pi \int_a^b (f(x))^2 \, dx.}$$

π $\quad$ r^2

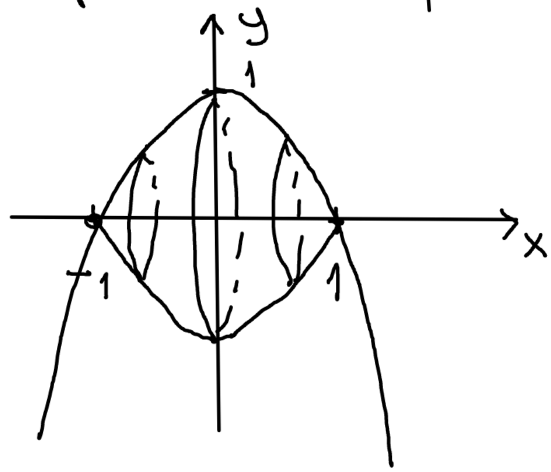


a) $y = 1 - x^2$

ničle: $1 - x^2 = 0$

$(1-x)(1+x) = 0$

$x_1 = 1 \qquad x_2 = -1$



$$V_x = \pi \cdot \int_{-1}^{1} (1 - x^2)^2 \, dx =$$

$$= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx =$$

$$= \pi \left(\left(x - 2 \cdot \frac{x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^{1} \right) =$$

$$= \pi \left(\left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right) =$$

$$= \pi \left(2 - \frac{4}{3} + \frac{2}{5} \right) = \pi \frac{30 - 20 + 6}{15} = \frac{16}{15}\pi$$

14. Izračunaj površine vrtenin, ki jih dobiš, če:

(a) krivuljo $y = {}^{+}\sqrt{4 - x^2}$ zavrtiš okrog x-osi med -2 in 2 in med -1 in 1.

(b) parabolo $y = x^2/2$ zavrtiš okrog x-osi med 0 in 1.

Pomagaj si s formulo

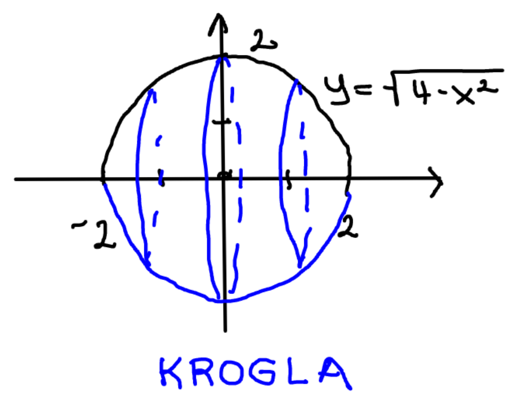
$$\boxed{P_x = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.}$$

a) $y = \sqrt{4-x^2} = (4-x^2)^{\frac{1}{2}} \quad /^2$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4 \xrightarrow{} r^2$

$y' = \frac{1}{\cancel{2}\sqrt{4-x^2}} \cdot (-\cancel{2}x) = \frac{-x}{\sqrt{4-x^2}}$



$P_x = 2\pi \cdot \int_{-2}^{2} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\, dx =$

$= 2\pi \int_{-2}^{2} \sqrt{4-x^2}\sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\, dx =$

$= 2\pi \int_{-2}^{2} \cancel{\sqrt{4-x^2}}\, \frac{\sqrt{4}}{\cancel{\sqrt{4-x^2}}}\, dx =$

$= 2\pi \int_{-2}^{2} 2\, dx = 4\pi \int_{-2}^{2} dx = 4\pi x \Big|_{-2}^{2} = 4\pi(2-(-2)) =$

$= 4\pi \cdot 4 = \underline{\underline{16\pi}}$

$P_{krogle} = 4\pi \cdot r^2 = 4\pi \cdot 2^2 = \underline{\underline{16\pi}}.$ ✓

$V_{krogle} = \frac{4\pi r^3}{3}$