Vaje MAT VSP, 2.12.2020

NEDOLOČENI INTEGRAL

Elementarni integrali

- $\int x^m \, dx = \frac{x^{m+1}}{m+1} + C$
- $\int e^x \, dx = e^x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \frac{dx}{\cos^2 x} = \tan + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
- $\int \frac{dx}{x} = \int x^{-1} \, dx = \log|x| + C$
- $\int \frac{dx}{1+x^2} = \arctan + C$

Pravili za računanje nedoločenih integralov

- $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$
- $\int \alpha f(x) \, dx = \alpha \int f(x) \, dx$

1. Izračunaj naslednje nedoločene integrale:

(a) $\int (3x^2 - 5x - \frac{1}{\sqrt{1-x^2}} + 1 - \cos x)dx$

(b) $\int (\sin x + \frac{2}{x^2} - \frac{1}{x})dx$

(c) $\int (x^6 - 2)^2\, dx$

(d) $\int (\frac{1}{\cos^2 x} - \frac{1}{1+x^2} + 5e^x)dx$

(e) $\int \sin(3x)dx$

(f) $\int \frac{dx}{5x-2}$

(g) $\int \frac{dx}{e^{2x}}$

(h) $\int \sin^4 x \cos x\, dx$

(i) $\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$

(j) $\int \frac{dx}{x \log^2(x)}$

(k) $\int (x^2 - 1)^9 x\, dx$

(l) $\int \tan x\, dx$

(m) $\int \frac{e^x}{e^x - 1}\, dx$

(n) $\int x e^{-(x^2+1)} dx$

(o) $\int \frac{x}{\cos^2(x^2)}\, dx$

(p) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

a) $\int (3x^2 - 5x - \frac{1}{\sqrt{1-x^2}} + 1 - \cos x)\, dx =$

$\left(\int 3x^2 dx = 3\int x^2 dx = \cancel{3} \cdot \frac{x^3}{\cancel{3}} + C = x^3 + C \right)$

$= x^3 - 5\frac{x^2}{2} - \arcsin x + x - \sin x + C$

$\int x^m dx = \frac{x^{m+1}}{m+1} + C$

$\left(\int -5x\, dx = -5\int x^1 dx = -5 \cdot \frac{x^2}{2} + C \right)$

b) $\int \left(\sin x + \frac{2}{x^2} - \frac{1}{x}\right) dx = -\cos x - \frac{2}{x} - \log|x| + C$

$$\left(\int \frac{2}{x^2} dx = 2\int \frac{1}{x^2} dx = 2\int x^{-2} dx = 2 \cdot \frac{x^{-1}}{-1} + C = -2 \cdot \frac{1}{x} + C \right)$$

c) $\int (x^6 - 2)^2 dx = \int (x^{12} - 4x^6 + 4)\, dx =$

$$= \frac{x^{13}}{13} - 4 \cdot \frac{x^7}{7} + 4x + C \qquad \left(\int 4 dx = 4\int dx = 4x + C \right)$$

Uvedba nove spremenljivke

e) $\int \sin (\underbrace{3x}_{u}) dx = \int \sin(u) \cdot \frac{du}{3} =$

$$u = 3x \;/'$$
$$du = 3\,dx \;/:3$$
$$\frac{du}{3} = dx$$

$$= \frac{1}{3} \int \sin u \; du = \frac{1}{3}(-\cos u) + C$$

$$= -\frac{1}{3} \cos(3x) + C$$

f) $\int \frac{dx}{5x-2} = \int \frac{\frac{du}{5}}{u} = \frac{1}{5}\int \frac{du}{u} =$

u

$$u = 5x - 2 \;/'$$
$$du = 5\,dx$$
$$\frac{du}{5} = dx$$

$$= \frac{1}{5} \log|u| + C$$

$$= \frac{1}{5} \log|5x-2| + C$$

g) $\int \frac{dx}{e^{2x}} = \int e^{-2x}\,\underline{dx} = \int e^{u} \cdot \frac{du}{-2} =$

$u = -2x$

$du = -2\,\underline{dx} \quad /:(-2)$

$\frac{du}{-2} = \underline{dx}$

$= -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + C = -\frac{1}{2}e^{-2x} + C$

h) $\int \sin^4 x \cdot \underline{\cos x\,dx} = \int u^4\,du = \frac{u^5}{5} + C$

$= \frac{\sin^5 x}{5} + C$

$u = \sin x$

$\underline{du} = \underline{\cos x\,dx}$

i) $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{u^2}{2} + C =$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{\arcsin^2 x}{2} + C$

j) $\int \frac{dx}{x\log^2 x} = \int \frac{du}{u^2} = \int u^{-2}\,du =$

$u = \log x \;/\,'$

$du = \frac{1}{x}\,dx$

$= \frac{u^{-1}}{-1} + C$

$= -\frac{1}{u} + C$

$= -\frac{1}{\log x} + C$

k) $\int (x^2-1)^9 \underbrace{x\,dx} = \int u^9 \frac{du}{2} = \frac{1}{2}\int u^9\,du =$

$u = x^2 - 1$

$du = 2\underbrace{x\,dx} \;/ : 2$

$\frac{du}{2} = \underbrace{x\,dx}$

$= \frac{1}{2}\,\frac{u^{10}}{10} + C$

$= \frac{(x^2-1)^{10}}{20} + C$

e) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} =$

$u = \cos x$

$du = -\underbrace{\sin x\,dx}$

$= -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C$

p) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^{u}\, 2\, du = 2\int e^{u}\, du =$

$= 2e^{u} + C$

$= 2e^{\sqrt{x}} + C$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$du = \frac{1}{2\sqrt{x}}dx \quad \cdot 2$

$2\, du = \frac{dx}{\sqrt{x}}$

2. Izračunaj naslednje nedoločene integrale z uporabo metode *per partes*:

(a) $\int x \log x \, dx$

(b) $\int (2x-1) \sin x \, dx$

(c) $\int \arctan(x) \, dx$

(d) $\int \arcsin(2x) \, dx$

PER PARTES (integracija po delih):

$$\int u \, dv = u \cdot v - \int v \, du$$

u – tisto, kar znamo odvajati

dv – tisto, kar znamo integrirati

$$\left[\begin{array}{lcl} u = & \xrightarrow{\ '\ } & du = \\ dv = \underline{\quad}\, dx & \xrightarrow{\int} & v = \end{array}\right]$$

a) $$\int \underbrace{x}_{dv}\,\underbrace{\log x}_{u}\, dx =$$

$$\left[\begin{array}{lcl} u = \log x & \xrightarrow{\ '\ } & du = \dfrac{1}{x}\, dx \\ dv = x\, dx & \xrightarrow{\ \int\ } & v = \dfrac{x^2}{2} \end{array}\right]$$

$$= \log x \cdot \frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\, dx =$$

$$= \frac{x^2}{2} \cdot \log x - \frac{1}{2}\int x\, dx =$$

$$= \frac{x^2}{2} \log x - \frac{1}{2}\,\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + C$$

b) $$\int (2x-1) \sin x\, dx =$$

$$\left[\begin{array}{lcl} u = 2x-1 & \xrightarrow{\ '\ } & du = 2\, dx \\ dv = \sin x\, dx & \xrightarrow{\ \int\ } & v = -\cos x \end{array}\right]$$

$$= -(2x-1)\cos x - \int (-\cos x) \cdot 2\, dx =$$

$$= -(2x-1)\cos x + 2\int \cos x\, dx =$$

$$= -(2x-1)\cos x + 2 \sin x + C$$

$$\int dx = x$$

d) $\int \arcsin(2x)\,dx =$

$$\left[\begin{array}{ll} u = \arcsin(2x) \xrightarrow{\ '\ } & du = \frac{1}{\sqrt{1-4x^2}} \cdot 2\,dx \\ dv = dx \xrightarrow{\ \int\ } & v = x \end{array}\right]$$

$$= x \cdot \arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}}\,dx =$$

nova spremenljivka

$$u = \sqrt{1-4x^2}$$

$$du = \frac{1}{\cancel{2}\sqrt{1-4x^2}} \cdot (-\cancel{8}x)\,dx$$

$$du = -\frac{4x}{\sqrt{1-4x^2}}\,dx \quad /(:-2)$$

$$-\frac{du}{2} = \frac{2x}{\sqrt{1-4x^2}}\,dx$$

$$= x \arcsin(2x) + \int \frac{du}{2}$$

$$= x \arcsin(2x) + \frac{1}{2}\int du$$

$$= x \arcsin(2x) + \frac{1}{2} u + C$$

$$= x \arcsin(2x) + \frac{\sqrt{1-4x^2}}{2} + C$$

3. Izračunaj nedoločene integrale naslednjih racionalnih funkcij.

(a) $\int \frac{x+6}{(x-1)(x-8)}\,dx$

(b) $\int \frac{x^2}{x+1}\,dx$

(c) $\int \frac{x+3}{x-3}\,dx$

(d) $\int \frac{x^2-1}{x^2+1}\,dx$

(e) $\int \frac{x^3+1}{x^2+4}\,dx$

(f) $\int \frac{2x^3+5x}{x^4+5x^2-1}\,dx$

a) $\int \frac{x+6}{(x-1)(x-8)}\,dx$

Razcep na parcialne ulomke:

$$\boxed{\frac{x+6}{(x-1)(x-8)}} = \frac{A}{x-1} + \frac{B}{x-8} = \boxed{\frac{A(x-8)+B(x-1)}{(x-1)(x-8)}}$$

$$x+6 = A(x-8)+B(x-1)$$
$$x+6 = Ax - 8A + Bx - B$$
$$x+6 = (A+B)x - 8A - B$$

$$1 = A+B$$
$$6 = -8A-B \quad \oplus$$
$$7 = -7A$$
$$\boxed{A=-1}$$

$$\frac{x+6}{(x-1)(x-8)} = \frac{-1}{x-1} + \frac{2}{x-8}$$

$$1 = -1+B$$
$$\boxed{B=2}$$

$$\int \frac{x+6}{(x-1)(x-8)}\,dx = \int \left(-\frac{1}{x-1} + \frac{2}{x-8}\right) dx \;=$$

$$\left(-\int \frac{1}{x-1}\,dx = -\int \frac{1}{u}\,du = -\log|u| + C = -\log|x-1| + C \right)$$

$u = x-1$
$du = dx$

$$\boxed{\int \frac{a}{x+b}\,dx = a\cdot \log|x+b| + C}$$

$$= -\log|x-1| + 2\log|x-8| + C$$

b) $\displaystyle\int \frac{x^2}{x+1}\,dx$

če je stopnja polinoma v števcu višja kot stopnja polinoma v imenovalcu, delimo polinoma

$$x^2 : (x+1) = x - 1$$

$$\begin{array}{l} \oplus\; -x^2 \pm x \\ \hline \quad -x \\ \oplus\; +x+1 \\ \hline \qquad 1 \end{array}$$

$$\frac{x^2}{x+1} = (x-1) + \frac{1}{x+1}$$

$(x-1)$ → celi del

1 → ostanek

$$\int \frac{x^2}{x+1}\,dx = \int \left(x-1+\frac{1}{x+1}\right)dx =$$

$$= \frac{x^2}{2} - x + \log|x+1| + C$$

d) $\int \frac{x^2-1}{x^2+1}\,dx$

enaki stopnji polinomov v števcu in v imenovalcu, lahko delimo

$$(x^2-1):(x^2+1) = 1$$

$$\begin{array}{r} -x^2 - 1 \\ \hline -2 \end{array}$$

$$\boxed{\frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}}$$

$$\int \frac{x^2-1}{x^2+1}\,dx = \int \left(1 - \frac{2}{x^2+1}\right)dx =$$

$$= x - 2\arctan x + C$$

Določeni integral

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

4. b) $$\int_0^{\frac{\pi}{3}} \tan x \; dx = -\log|\cos x|\Big|_0^{\frac{\pi}{3}} =$$

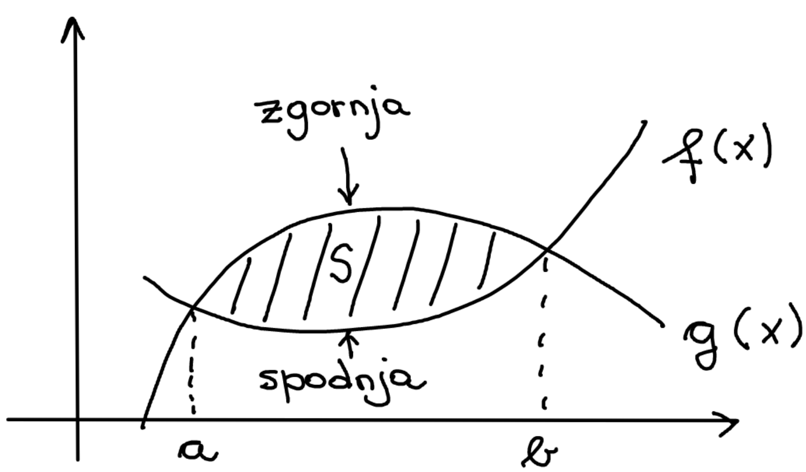
$$= -\log\left|\cos\frac{\pi}{3}\right| - \left(-\log|\cos 0|\right)$$

$$= -\log\left(\frac{1}{2}\right) + \log 1 = -\log\left(\frac{1}{2}\right) =$$

$$= -\log\left(2^{-1}\right) =$$

$$= \log(2)$$

Ploščina območja med f in g:



1. Izračunamo presečišča: $f(x) = g(x)$,
2. Presečišči a in b $\Rightarrow$ ploščina: $$S = \int_a^b \Big(\underset{\substack{\downarrow\\ \text{zgornja}}}{g(x)} - \underset{\substack{\downarrow\\ \text{spodnja}}}{f(x)}\Big)\,dx$$