Vaje MAT VSP, 5.11.2020

VRSTE

Vrsta je vsota $a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$.

Tvorimo zaporedje S_n delnih vsot:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

Velja: $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$

Vrsta $\sum_{n=1}^{\infty} a_n$ je konvergentna (ima končno vsoto, če je S_n konvergentno.

$$\lim_{n \to \infty} a_n = 0$$

Če $\lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ ni konvergentna.

Če $\lim_{n \to \infty} a_n = 0$, pa vrsta $\sum_{n=1}^{\infty} a_n$ še ni nujno konvergentna.

NALOGA 22. *OR*

Naj bo (a_n) zaporedje $a_n = \frac{1}{n(n+1)}$.

a. Poišči $\lim_{n\to\infty} a_n$.
b. S formulo izrazi N-to delno vsoto vrste

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

tj. $S_N = a_1 + a_2 + \cdots + a_N$. (*Namig:* Zapiši $\frac{1}{n(n+1)}$ kot vsoto parcialnih ulomkov.)
c. Seštej zgornjo vrsto; izračunaj limito delnih vsot $\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n(n+1)} = 0.$

b) $a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \quad / \cdot n(n+1)$

$$1 = A(n+1) + B\cdot n$$
$$1 = An + A + Bn$$
$$0\cdot n + 1 = (A+B)n + A$$

$$0 = A+B \qquad 1 = A$$
$$A = -B \qquad B = -1$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$S_1 = a_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$
$$= 1 - \frac{1}{4} = \frac{3}{4}$$

$$\boxed{S_n = \frac{n}{n+1}}$$

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n =$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) +$$

$$+ \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

c) $$\sum_{n=1}^{\infty} a_n = \sum_{n=1} \frac{1}{n(n+1)} = \lim_{n \to \infty} S_n =$$

$$= \lim_{n \to \infty} \frac{n \;/:n}{n+1 \;/:n} =$$

$$= \lim_{n \to \infty} \frac{1}{1 + \underbrace{\frac{1}{n}}_{\to 0}} = \frac{1}{1} = 1$$

NALOGA 23. *OR*

Izračunaj vsote naslednjih geometrijskih vrst:

a. $\sum_{n=0}^{\infty} \frac{1}{4^n}$

b. $\sum_{n=1}^{\infty} \frac{10}{3^n}$

c. $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}}$

d. $\frac{3}{2} + 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots$

e. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}}$

f. $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^{3n}$, za tiste $x \in \mathbb{R}$, za katere vrsta konvergira.

Geometrijska vrsta

$$a_1 + a_1 \cdot q + a_1 \cdot q^2 + \ldots = \sum_{n=0}^{\infty} a_1 \cdot q^n, \quad q = \frac{a_{n+1}}{a_n}$$

Konvergenca je odvisna od kvocienta q:

- $|q|<1$, vrsta konvergira
- $|q|\geq 1$, vrsta divergira

$$\boxed{\text{Za } |q|<1 \qquad \sum_{n=0}^{\infty} a_1 \cdot q^n = \frac{a_1}{1-q}}$$

a) $$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1} + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{1}{1-\frac{1}{4}} =$$

$\cdot\frac{1}{4}$ $\quad \cdot\frac{1}{4}$

$$= \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$a_1 = 1$

$q = \frac{a_2}{a_1} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$

b) $$\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10}{3} + \frac{10}{3^2} + \frac{10}{3^3} + \dots = \frac{\frac{10}{3}}{1-\frac{1}{3}} =$$

$\cdot\frac{1}{3}$ $\quad \frac{1}{3}$

$$= \frac{\frac{10}{3}}{\left[\frac{2}{3}\right]} =$$

$a_1 = \frac{10}{3}$

$q = \frac{1}{3}$

$$= \frac{10\cdot\cancel{3}}{\cancel{3}\cdot 2} = 5$$

c) $$\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}} = \frac{2^2}{3^3} + \frac{2^3}{3^5} + \frac{2^4}{3^7} + \dots = \frac{\frac{4}{27}}{1-\frac{2}{9}} =$$

$\cdot\frac{2}{3^2}$ $\quad \cdot\frac{2}{3^2}$

$$= \frac{\frac{4}{27}}{\frac{7}{9}} =$$

$a_1 = \frac{4}{27}$

$q = \frac{2}{9}$

$$= \frac{4\cdot\cancel{9}}{3\,\cancel{27}\cdot 7} = \frac{4}{21}$$

d) $\frac{3}{2}+1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots=\frac{\frac{3}{2}}{1-\frac{2}{3}}=\frac{\frac{3}{2}}{\frac{1}{3}}=$

$\cdot\frac{2}{3} \quad \cdot\frac{2}{3}$

$a_1=\frac{3}{2}$ $\qquad =\frac{3\cdot 3}{2\cdot 1}=\underline{\underline{\frac{9}{2}}}$

$q=\frac{2}{3}$

e) $\sum_{n=1}^{\infty}\frac{(-2)^n}{3\cdot 2^{3n-2}}=\frac{-2}{3\cdot 2}+\frac{(-2)^2}{3\cdot 2^4}+\frac{(-2)^3}{3\cdot 2^7}+\dots=$

$\cdot\frac{-2}{2^3}$

$a_1=\frac{-\cancel{2}}{3\cdot\cancel{2}}=-\frac{1}{3}$

$q=-\frac{2}{2^3}=-\frac{1}{4}$ $\qquad =\frac{-\frac{1}{3}}{1+\frac{1}{4}}=\frac{-\frac{1}{3}}{\frac{5}{4}}=-\frac{4\cdot 1}{3\cdot 5}$

$=-\underline{\underline{\frac{4}{15}}}$

f) $\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{3n}=\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^6+\left(\frac{x}{2}\right)^9+\dots$

$\cdot\left(\frac{x}{2}\right)^3 \quad \cdot\left(\frac{x}{2}\right)^3$

$a_1=\left(\frac{x}{2}\right)^3$

$q=\left(\frac{x}{2}\right)^3$ $\qquad =\frac{\left(\frac{x}{2}\right)^3}{1-\left(\frac{x}{2}\right)^3}=\frac{\frac{x^3}{8}}{1-\frac{x^3}{8}}=$

$=\frac{\frac{x^3}{8}}{\frac{8-x^3}{8}}=\frac{\cancel{8}x^3}{\cancel{8}(8-x^3)}=\frac{x^3}{8-x^3}$

Za katere $x \in \mathbb{R}$ vrsta konvergira?

$$\left|\left(\frac{x}{2}\right)^3\right| < 1$$

$x \geq 0$

$$\left(\frac{x}{2}\right)^3 < 1 \quad /\sqrt[3]{}$$

$$\frac{x}{2} < 1 \quad / \cdot 2$$

$x < 2$

$0 \leq x < 2$

$x < 0$

$$-\left(\frac{x}{2}\right)^3 < 1 \quad / \cdot (-1)$$

$$\left(\frac{x}{2}\right)^3 > (-1) \quad /\sqrt[3]{}$$

$$\frac{x}{2} > -1 \quad / \cdot 2$$

$x > -2$

$-2 < x < 0$

$-2 < x < 2$

NALOGA 24. *OR*

Kateri racionalni ulomek ima decimalni zapis $0.\overline{12} = 0.121212\ldots$? Pomagaj si s primerno geometrijsko vrsto.

$$x = 0,\overline{12} \quad / \cdot 100$$

$$100x = 12,\overline{12}$$

$$99x = 12$$

$$x = \frac{12}{99} = \frac{4}{33}$$

Z uporabo geometrijske vrste:

$$0,\overline{12} = \underbrace{0,12}\underbrace{12}\underbrace{12}\ldots = 0,12 + 0,0012 + 0,000012 + \ldots$$

$$= \frac{12}{10^2} + \frac{12}{10^4} + \frac{12}{10^6} + \ldots =$$

$$\cdot \frac{1}{10^2} \qquad \frac{1}{10^2}$$

$$a_1 = \frac{12}{10^2}$$

$$q = \frac{1}{10^2}$$

$$= \frac{\frac{12}{10^2}}{1 - \frac{1}{10^2}} = \frac{\frac{12}{10^2}}{\frac{99}{10^2}} = \frac{12 \cdot \cancel{10}^2}{\cancel{10}^2 \cdot 99} =$$

$$= \frac{12}{99} = \frac{4}{33}$$

FUNKCIJE

$$f: D_f \to \mathbb{R}, \quad D_f \subseteq \mathbb{R}$$
$$x \mapsto f(x)$$

Graf funkcije $f: D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}$

$$\Gamma(f) = \{ (x, f(x)); \ x \in D_f \} \subseteq \mathbb{R} \times \mathbb{R}$$

gama

NALOGA 26. *OR*

Poišči predpise za inverze, $f^{-1}(x)$, spodnjih funkcij. Na katerih območjih v $\mathbb{R}$ imajo ti predpisi smisel?

a. $f(x) = \frac{x+1}{2x-3}$,

b. $g(x) = \frac{2x}{x^2+1}$,

c. $h(x) = \log(2x-1)$,

d. $k(x) = \sqrt{x^2+1}$.

$$f, g, h, k: \mathbb{R} \to \mathbb{R}$$

Funkcija $f: D_f \to \mathbb{R}$ je INJEKTIVNA, če

$$\boxed{\text{za vsak } x, y \in D_f : x \neq y \Rightarrow f(x) \neq f(y).}$$

$\Updownarrow$ (različni točki se slikata v različni vrednosti)

$$\boxed{\text{za vsak } x, y \in D_f : f(x) = f(y) \Rightarrow x = y}$$

Vsaka vodoravna premica seka graf injektivne funkcije v največ eni točki.

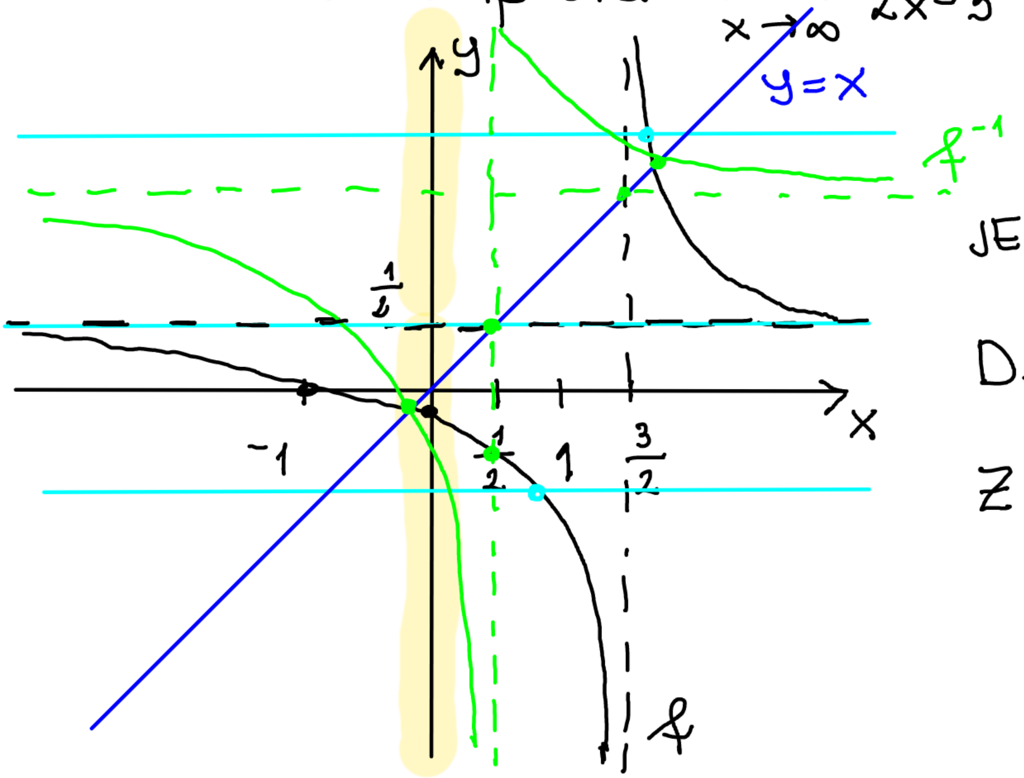
a) $f(x) = \dfrac{x+1}{2x-3}$

ničle: $x + 1 = 0$
$x = -1$

poli: $2x - 3 = 0$
$2x = 3$
$x = \frac{3}{2}$

zač. vrednost: $f(0) = \dfrac{0+1}{2 \cdot 0 - 3} = -\dfrac{1}{3}$

vodoravna asimptota: $\lim\limits_{x \to \infty} \dfrac{x+1}{2x-3} = \dfrac{1}{2}$



JE INJEKTIVNA

$D_f : \mathbb{R} \setminus \{\frac{3}{2}\}$

$Z_f : \mathbb{R} \setminus \{\frac{1}{2}\}$

INJEKTIVNOST: $x_1, x_2 \in D_f$

$$\boxed{f(x_1) = f(x_2)}$$

$$\frac{x_1+1}{2x_1-3} = \frac{x_2+1}{2x_2-3} \quad / \cdot (2x_1-3)(2x_2-3)$$

$$(x_1+1)(2x_2-3) = (x_2+1)(2x_1-3)$$

$$\cancel{2x_1x_2} - 3x_1 + 2x_2 - \cancel{3} = \cancel{2x_1x_2} - 3x_2 + 2x_1 - \cancel{3}$$

$$-3x_1 + 2x_2 = -3x_2 + 2x_1$$

$$-3x_1 - 2x_1 = -3x_2 - 2x_2$$

$$-5x_1 = -5x_2 \quad / : (-5)$$

$$\boxed{x_1 = x_2}$$

JE INJEKTIVNA

INVERZ: $f(x) = y = \dfrac{x+1}{2x-3}$

zamenjamo x in y

$$x = \frac{y+1}{2y-3} \quad / (2y-3)$$

$$x(2y-3) = y+1$$

$$2xy - 3x = y+1$$

$$2xy - y = 3x+1$$

$$y(2x-1) = 3x+1 \quad / : (2x-1)$$

$$f^{-1}(x) = y = \frac{3x+1}{2x-1}$$

$2x-1=0$
$x=\frac{1}{2}$

ima smisel za $\mathbb{R} \setminus \{\frac{1}{2}\}$

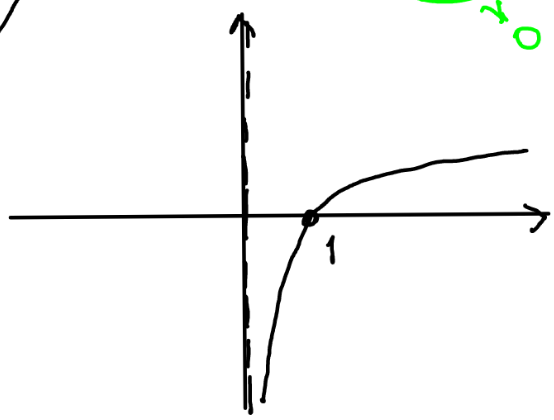
Graf inverzne funkcije f^{-1} dobimo tako, da graf funkcije f zrcalimo preko simetrale lihih kvadrantov ($y=x$).

$D_{f^{-1}} = Z_f = \mathbb{R}\setminus\left\{\frac{1}{2}\right\}$

$Z_{f^{-1}} = D_f = \mathbb{R}\setminus\left\{\frac{3}{2}\right\}$

c) $h(x) = \log(2x-1)$ $\quad h: \mathbb{R} \to \mathbb{R}$

> 0



$\log x$, $D_{\log x}: x > 0$

ničle: $\log_e(2x-1) = 0$

$e^0 = 2x-1$

$1 = 2x-1$

$2 = 2x \quad | :2$

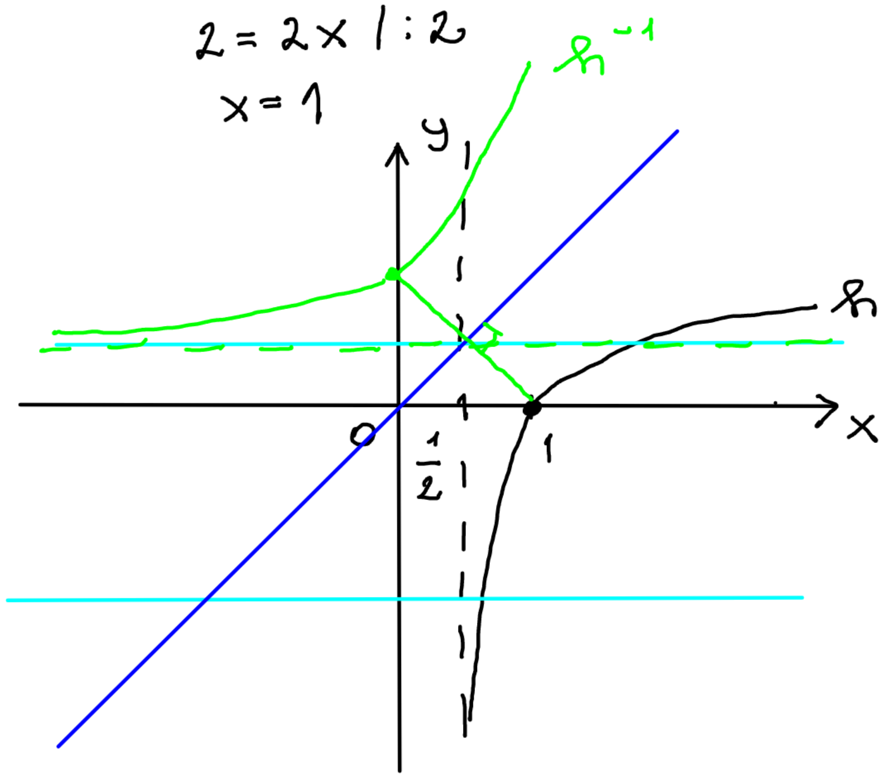
$x = 1$

$D_h: 2x-1 > 0$

$2x > 1$

$x > \frac{1}{2}$

$Z_h: \mathbb{R}$

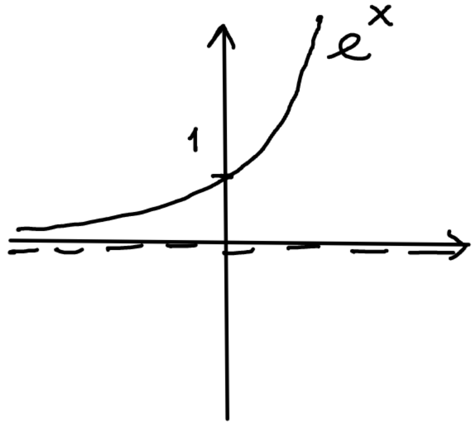


JE INJEKTIVNA

$$h(x) = y = \log(2x-1)$$

INVERZ: $x = \log(2y-1)$

$$e^x = e^{\log(2y-1)}$$



$$e^x = 2y-1$$

$$e^x + 1 = 2y \quad /:2$$

$$h^{-1}(x) = y = \frac{e^x+1}{2}$$

ima smiscl za vse $x \in \mathbb{R}$

$D_{h^{-1}}: \mathbb{R}$
$Z_{h^{-1}}: \left(\frac{1}{2}, \infty\right)$