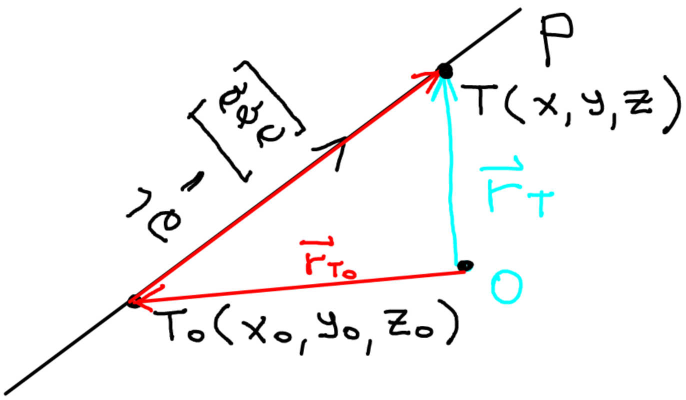
Vaje MAT VSP, 23.12.2020

PREMICE IN RAVNINE

Premice



$$\boxed{\vec{r}_T = \vec{r}_{T_0} + t \cdot \vec{s}, \quad t \in \mathbb{R}}$$

vektorska oblika enačbe premice

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$t = \frac{x - x_0}{a} \leftarrow \quad x = x_0 + ta$$

$$t = \frac{y - y_0}{b} \leftarrow \quad y = y_0 + tb \qquad t \in \mathbb{R}$$

$$t = \frac{z - z_0}{c} \leftarrow \quad z = z_0 + tc$$

parametrična oblika

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

kanonična oblika

1. Dane so točke $A(3,2,0)$, $B(2,1,2)$ in $C(4,1,6)$.

(a) Določi premico p skozi točki A in B. Premico zapiši v parametrični in ~~implicitni~~ kanonični obliki.

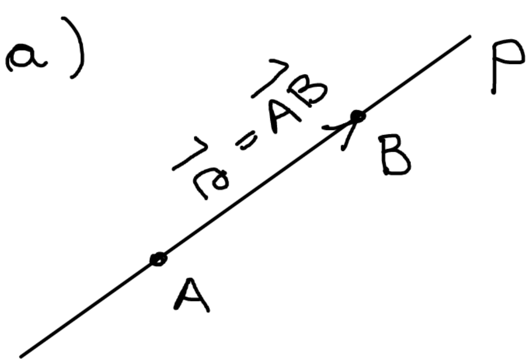
(b) Ali so točke A, B in C kolinearne?

(c) Poišči točko D na premici p, tako da bo vektor $\overrightarrow{CD}$ pravokoten na p. Nato določi razdaljo med točko C in premico p.

(d) Poišči zrcalno sliko C' pri zrcaljenju točke C čez premico p.

(e) Poišči točki P, Q na premici p, tako da bo $CPC'Q$ kvadrat.

a)



$$\vec{r} = \vec{r}_A + t \cdot \overrightarrow{AB}$$

$$\boxed{\vec{r} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}}$$

vektorska oblika

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A = $$

$$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 3 - t \\ y &= 2 - t \qquad t \in \mathbb{R} \\ z &= 0 + 2t \end{aligned}}$$

parametrična oblika

$$\boxed{\frac{x-3}{-1} = \frac{y-2}{-1} = \frac{z-0}{2}}$$

kanonična oblika

b) A, B, C kolinearne?

$\downarrow$

Ali ležijo na isti premici?

$\downarrow$

Ali C leži na p?

Vstavimo točko C v kanonično obliko premice p:

$$C(\overset{x}{4}, \overset{y}{1}, \overset{z}{6})$$

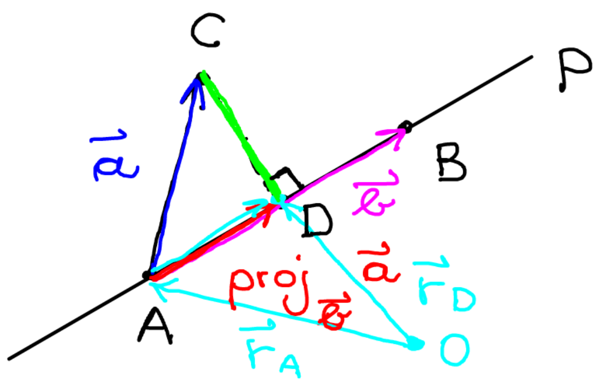
$$\frac{x-3}{-1} = \frac{y-2}{-1} = \frac{z-0}{2}$$

$$\frac{4-3}{-1} = \frac{1-2}{-1} = \frac{6}{2}$$

$$-1 = 1 = 3 \quad //$$

Sledi, C ne leži na premici p.

(c) Poišči točko D na premici p, tako da bo vektor $\overrightarrow{CD}$ pravokoten na p. Nato določi razdaljo med točko C in premico p.



$$\vec{a} = \overrightarrow{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$$\vec{b} = \overrightarrow{AB} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \cdot \vec{b} = \frac{12}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix}$$

$$\vec{a}\cdot\vec{b} = \begin{bmatrix}1\\-1\\6\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \cancel{1(-1)} + \cancel{(-1)(-1)} + 6\cdot 2 = 12$$

$$|\vec{b}| = \sqrt{(-1)^2+(-1)^2+2^2} = \sqrt{6}$$

$$\vec{r}_D = \vec{r}_A + \mathrm{proj}_{\vec{b}}\,\vec{a} = \begin{bmatrix}3\\2\\0\end{bmatrix} + \begin{bmatrix}-2\\-2\\4\end{bmatrix} = \begin{bmatrix}1\\0\\4\end{bmatrix}$$

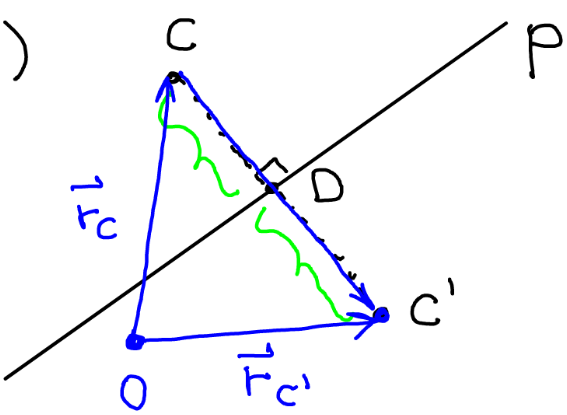
$$\boxed{D(1,0,4)}$$

Razdalja točke C do premice p je $|\overrightarrow{CD}|$ (dolžina vektorja $\overrightarrow{CD}$).

$$\overrightarrow{CD} = \vec{r}_D - \vec{r}_C = \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}4\\1\\6\end{bmatrix} = \begin{bmatrix}-3\\-1\\-2\end{bmatrix}$$

$$|\overrightarrow{CD}| = \sqrt{(-3)^2+(-1)^2+(-2)^2} = \underline{\underline{\sqrt{14}}}$$

d)



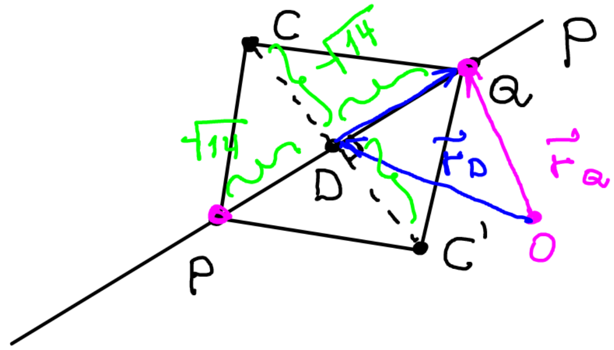
$$\vec{r}_{C'} = \vec{r}_C + 2\overrightarrow{CD} =$$

$$= \begin{bmatrix}4\\1\\6\end{bmatrix} + 2\begin{bmatrix}-3\\-1\\-2\end{bmatrix} =$$

$$= \begin{bmatrix}4\\1\\6\end{bmatrix} + \begin{bmatrix}-6\\-2\\-4\end{bmatrix} = \begin{bmatrix}-2\\-1\\2\end{bmatrix}$$

$$\boxed{C'(-2,-1,2)}$$

(e) Poišči točki P, Q na premici p, tako da bo $CPC'Q$ kvadrat.



vektor v smeri $\vec{a}$, dolžine 1

$$\vec{r}_Q = \vec{r}_D + \sqrt{14}\,\frac{\vec{a}}{|\vec{a}|}$$

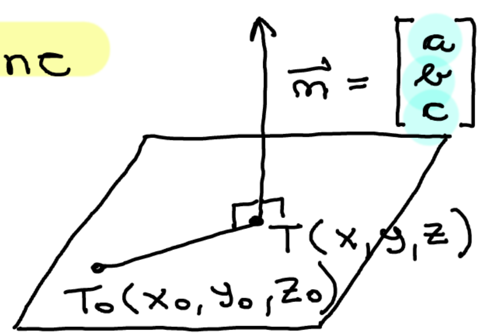
$$\vec{r}_P = \vec{r}_D - \sqrt{14}\,\frac{\vec{a}}{|\vec{a}|}$$

$$\vec{r}_Q = \begin{bmatrix}1\\0\\4\end{bmatrix} + \frac{\sqrt{14}}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix} \qquad \vec{r}_P = \begin{bmatrix}1\\0\\4\end{bmatrix} - \frac{\sqrt{14}}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\vec{a} = \vec{AB} = \begin{bmatrix}-1\\-1\\2\end{bmatrix} \qquad |\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$$

Ravnine

$$\vec{n} = \begin{bmatrix}a\\b\\c\end{bmatrix} \text{ normala}$$



$$\vec{n} \perp \vec{T_0T}$$

$$\vec{n} \cdot \underbrace{\vec{T_0T}}_{\vec{r}_T - \vec{r}_{T_0}} = 0$$

$$\begin{bmatrix}a\\b\\c\end{bmatrix} \cdot \begin{bmatrix}x-x_0\\y-y_0\\z-z_0\end{bmatrix} = 0$$

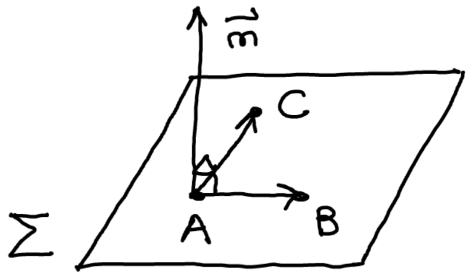
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax-ax_0+by-by_0+cz-cz_0=0$$

$$ax+by+cz=ax_0+by_0+cz_0$$

$$\boxed{\vec{n}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{n}\cdot\vec{r}_{T_0}}$$

enačba ravnine



$$\vec{m} = \vec{AB} \times \vec{AC}$$

3. Dane so točke $A(2,3,1)$ $B(1,-1,1)$, $C(2,1,3)$ in $D(9,0,-4)$.

   (a) Določi enačbo ravnine Σ, ki gre skozi točke A, B in C.

   (b) Poišči ravnino skozi točko D, ki je vzporedna ravnini Σ.

   (c) Določi razdaljo med ravnino Σ in točko D. Poišči še zrcalno sliko D' pri zrcaljenju točke D čez Σ.

a) $$\vec{m} = \vec{AB} \times \vec{AC} = \begin{bmatrix}-1\\4\\0\end{bmatrix} \times \begin{bmatrix}0\\-2\\2\end{bmatrix} = \begin{bmatrix}-8\\2\\2\end{bmatrix} = 2\underbrace{\begin{bmatrix}-4\\1\\1\end{bmatrix}}_{\vec{m}}$$

$$\vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\-4\\0\end{bmatrix}$$

$$\vec{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}0\\-2\\2\end{bmatrix}$$

pomembna je le smer $\vec{m}$, dolžina pa ne

Enačba ravnine: $$\vec{m} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{m} \cdot \vec{r}_A$$

$$\begin{bmatrix}-4\\1\\1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-4\\1\\1\end{bmatrix}\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$-4x + 1y + 1z = (-4)2 + 1\cdot 3 + 1\cdot 1$$

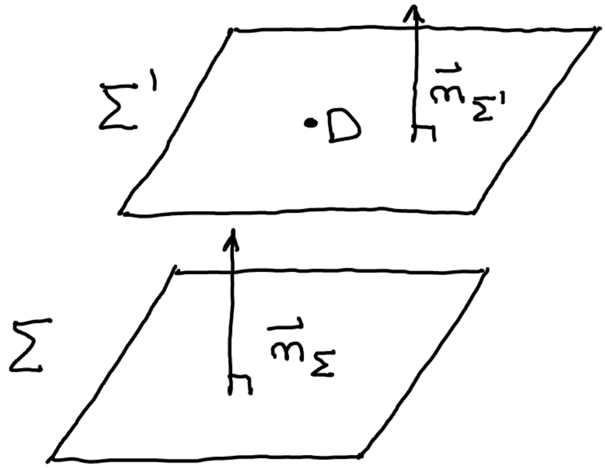
$$\Sigma: \boxed{-4x + y + z = -4}$$

Ali D leži na Σ?
$D(9,0,-4)$

$$-4\cdot 9 + 0 - 4 = -4$$
$$-40 = -4 \quad //$$

D ne leži na ravnini Σ

b)



$$\vec{m}_{\Sigma'} = \vec{m}_{\Sigma}$$

$$\vec{m}_{\Sigma'} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

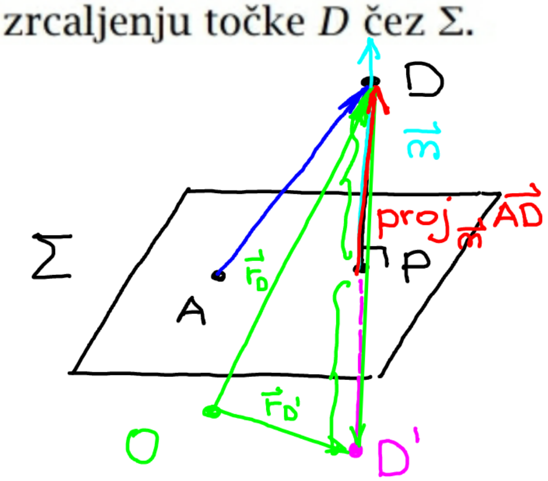
$$\vec{m}_{\Sigma'} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{m}_{\Sigma'} \cdot \vec{r}_D$$

$$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix}$$

$$-4x + 1 \cdot y + 1 \cdot z = -4 \cdot 9 + 1 \cdot 0 + 1(-4)$$

$$\Sigma' : \boxed{-4x + y + z = -40}$$

(c) Določi razdaljo med ravnino Σ in točko D. Poišči še zrcalno sliko D' pri zrcaljenju točke D čez Σ.



Razdalja med Σ in D je $|\vec{PD}|$

$$|\vec{PD}| = |\text{proj}_{\vec{m}} \vec{AD}|$$

$$\text{proj}_{\vec{m}} \vec{AD} = \frac{\vec{m} \cdot \vec{AD}}{|\vec{m}|^2} \cdot \vec{m} = \frac{-36}{18} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$$

$$\vec{AD} = \vec{r}_D - \vec{r}_A = \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$$

$$\vec{m} \cdot \vec{AD} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix} = (-4) \cdot 7 + 1(-3) + 1(-5) = -36$$

$$|\vec{m}| = \sqrt{(-4)^2 + 1^2 + 1^2} = \sqrt{18}$$

Razdalja med D in Σ:

$$|\vec{PD}| = |\text{proj}_{\vec{m}} \vec{AD}| = \sqrt{8^2 + (-2)^2 + (-2)^2} = \underline{\underline{\sqrt{72}}}$$

Zrcalna slika D':

$$\vec{r}_{D'} = \vec{r}_D + 2 \cdot \underbrace{\vec{DP}}_{\substack{\shortparallel \\ -\vec{PD}}} = \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix} - 2\,\vec{PD} = \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix} - 2 \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 16 \\ -4 \\ -4 \end{bmatrix} =$$

$$= \begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix}$$

$$\boxed{D'(-7, 4, 0)}$$

Sistemi linearnih enačb

7. Z uporabo Gaussove eliminacije poišči vse rešitve naslednjih sistemov linearnih enačb:

(a)
$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_4 &= 4 \\ 4x_1 + 3x_2 - x_3 + 2x_4 &= 6 \\ 8x_1 + 5x_2 - 3x_3 + 4x_4 &= 12 \\ 3x_1 + 3x_2 - 2x_3 + 2x_4 &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 7x_2 + 3x_3 + x_4 &= 5 \\ x_1 + 3x_2 + 5x_3 - 2x_4 &= 3 \\ x_1 + 5x_2 - 9x_3 + 8x_4 &= 1 \\ 5x_1 + 18x_2 + 4x_3 + 5x_4 &= 12 \end{aligned}$$

(c)
$$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 8 \\ 3x_1 - 2x_2 + x_3 - 3x_4 &= 7 \\ 2x_1 - x_2 - 5x_4 &= 6 \\ 5x_1 - 3x_2 + x_3 - 8x_4 &= 1 \end{aligned}$$

(d)
$$\begin{aligned} 3x_1 + 4x_2 + x_3 + 2x_4 &= 3 \\ 6x_1 + 8x_2 + 2x_3 + 5x_4 &= 7 \\ 9x_1 + 12x_2 + 3x_3 + 10x_4 &= 13 \end{aligned}$$

a) $(-2)\cdot$ ⊕ 4

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 2 & 6 \\ 8 & 5 & -3 & 4 & 12 \\ 3 & 3 & -2 & 2 & 6 \end{array}\right] \begin{array}{c} (-4) \\ \sim \\ + \\ \end{array} \quad (-3) \quad 2 \quad +$$

pod diagonalo bi radi imeli ničle

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} /(-3) \\ \sim \\ ⊕ \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \sim \\ ⊕ \\ /(-2) \end{array}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right]$$

Dovoljene operacije:

- Lahko zamenjamo dve vrstici med seboj.
- Posamezni vrstici lahko prištejemo večkratnik neke druge vrstice.
- Vrstico lahko pomnožimo z realnim številom, različnim od 0.

$[0\ \ 0\ \ 0 \dots 0 \mid 0]$

imamo neskončno rešitev

$[0\ \ 0\ \ 0\ \ 0 \dots \mid a]$, $a \neq 0$

nimamo rešitev $0 = a$ //

$$-2x_4 = 2 \quad |:(-2)$$
$$x_4 = -1$$

$$-2x_3 = 2 \quad |:(-2)$$
$$x_3 = -1$$

$$-x_2 + \overset{-1}{x_3} = -2$$
$$-x_2 - 1 = -2$$
$$-x_2 = -2 + 1$$
$$-x_2 = -1 \quad |:(-1)$$
$$x_2 = 1$$

$$2x_1 + 2\overset{1}{x_2} - \overset{-1}{x_3} + \overset{-1}{x_4} = 4$$

$$2x_1 + 2 \cdot 1 - \cancel{(-1)} + \cancel{(-1)} = 4$$

$$2x_1 = 2 \quad |:2$$
$$x_1 = 1$$