~~2 nal abs lin al (VP) li jedro st. Lred. inj., surj.~~

Primer: $S = \{ M \in \mathbb{R}^{2\times 2} ; M^T = M \}$

a) Pokažimo, da je S vektorski prostor in poiščimo kakšno bazo.

① Pokažimo, da je S VPP v $\mathbb{R}^{2\times 2}$.

$M, N \in S \Rightarrow M = M^T$ in $N = N^T$

$$(\alpha M + \beta N)^T = (\alpha M)^T + (\beta N)^T = \alpha(M^T) + \beta(N^T) = \alpha M + \beta N \in S$$

$\Rightarrow S$ je vektorski prostor.

② $M \in S \Rightarrow M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}$

Poiščimo take, ki so lin. neodv in lahko naredimo cel prost.

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} ; a, b, c \in \mathbb{R} \right\} = \mathcal{L}in\{E_{11}, E_{12}+E_{21}, E_{22}\}$$

Ker $S = \mathcal{L}in(\dots)$, je lahko tudi to dokaz, da je S VPP v $\mathbb{R}^{2\times 2}$.

Te tri matrike so lin. neodvisne, ker
- E_{11} ni lin. komb. $E_{12}+E_{21}$ in E_{22} (glej levi zgornji el.)
- $E_{12}+E_{21}$ ni lin. komb. ... (...)
- E_{22} ...

$\Rightarrow \mathcal{B} = \{E_{11}, E_{12}+E_{21}, E_{22}\}$ je baza S.

b) $\phi : S \to \mathbb{R}_2[x]$

$$\phi : \begin{bmatrix} a & b \\ b & c \end{bmatrix} \longmapsto ax^2 + bx + a + b$$

ϕ je linearna

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad N = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$$

$$\phi\left(\alpha \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \beta \begin{bmatrix} d & e \\ e & f \end{bmatrix}\right) = \phi\left(\begin{bmatrix} \alpha a + \beta d & \alpha b + \beta e \\ \alpha b + \beta e & \alpha c + \beta f \end{bmatrix}\right) =$$

$$= (\alpha a + \beta d)x^2 + (\alpha b + \beta e)x + (\alpha a + \beta d) + (\alpha b + \beta e) =$$

$$= \alpha(ax^2 + bx + a + b) + \beta(dx^2 + ex + d + e) =$$

$$= \alpha\, \phi\left(\begin{bmatrix} a & b \\ b & \boxed{c} \end{bmatrix}\right) + \beta\, \phi\left(\begin{bmatrix} d & e \\ e & f \end{bmatrix}\right) \Rightarrow \phi \text{ je linearna}$$

karkoli, ampak dokazujemo linearnost

c) Zapišimo matriko, ki pripada ϕ iz baze $\mathcal{B}$ (iz naloga) v $\mathcal{S} = \{1, x, x^2\}$.

Vzamemo elemente iz $\mathcal{B}$, preslikamo s ϕ, razvijemo po bazi $\mathcal{S}$.

$$\phi(E_{11}) = x^2 + 1 = \underline{1} \cdot 1 + \underline{0} \cdot x + \underline{1} \cdot x^2$$

$$\phi(E_{12} + E_{21}) = x + 1 = \underline{1} \cdot 1 + \underline{1} \cdot x + \underline{0} \cdot x^2$$

$$\phi(E_{22}) = 0 = \underline{0} \cdot 1 + \underline{0} \cdot x + \underline{0} \cdot x^2$$

$$A_{\phi, \mathcal{B}, \mathcal{S}} = \begin{array}{c c} \begin{array}{ccc} E_{11} & E_{12}+E_{21} & E_{22} \\ \downarrow & \downarrow & \downarrow \end{array} & \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

vedno zlagamo po stolpcih

d) Določimo $\ker\phi$.

$\ker \phi = \{ M \in S ; \ \phi(M) = 0 \} =$ ← ničelni polinom

$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} ; \ \underbrace{ax^2 + bx + a + b}_{a = b = a+b = 0} = 0 \right\} =$

$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} ; \ c \in \mathbb{R} \right\} = \mathcal{L}\{E_{22}\}$

e) Ali je ϕ injektivna?

$(\phi(M) = \phi(N) \Rightarrow M = N)$

Ni, ker ima jedro več kot element. Torej se dva različna elementa iz S preslikata v isti element. (primer: $\phi(E_{22}) = \phi(0)$)

V splošnem: $\tau : U \to V$ linearna preslikava.
Potem je τ injektivna natanko tedaj, ko $\ker(\tau) = \{0\}$.

Dokaz: ($\Rightarrow$) Pokazati želimo τ injektivna $\Rightarrow \ker\tau = \{0\}$
ali $\ker\tau \neq \{0\} \Rightarrow \tau$ NI injektivna

To je res, ker če $u \in \ker\tau$ in $u \neq 0_U$, potem $\tau(u) = \tau(0_U)$.

($\Leftarrow$) Pokazati želimo, da iz $\ker\tau = \{0_U\}$ sledi, da je τ injektivna.

Če $\tau(u_1) = \tau(u_2)$, potem

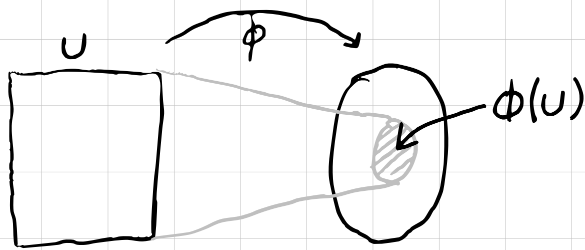
$\tau(u_1) - \tau(u_2) = 0_V$

$\tau(u_1 - u_2) = 0_V$ ker je τ linearna!

$u_1 - u_2 \in \ker\tau$

$u_1 - u_2 = 0_U$

$u_1 = u_2$



<u>Def:</u> $\underbrace{\tau(U)}_{\text{slika } \tau,\ \mathrm{im}(\tau)} = \{v \in V;\ v = \tau(u) \text{ za nek } u \in U\}.$

$\phi(U) = V \iff \phi$ surjektivna

f) Kako bi izračunali $\ker \phi$ s pomočjo $A_{\phi,\mathcal{B},\varphi}$?

$$A_{\phi,\mathcal{B},\varphi} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = aE_{11} + b(E_{12} + E_{21}) + cE_{22}$$

$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xleftrightarrow{\text{pripada}} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ vektor koeficientov v bazi $\mathcal{B}$

$$\phi\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$$

Jedro preslikave je torej ničelni prostor matrike.

$\tau: U \to V \qquad \ker \tau \leftrightarrow N(A)$

$$N(A_{\phi,\mathcal{B},\varphi}) = N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};\ x_1 = x_2 = 0\right\} = \left\{\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix};\ x_3 \in \mathbb{R}\right\} = \mathcal{L}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{G.el.}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \ker \phi = \{0 \cdot E_{11} + 0(E_{12} + E_{21}) + x_3 E_{22};\ x_3 \in \mathbb{R}\} = \mathcal{L}\{E_{22}\}$$

g) Kaj je $\operatorname{im} \phi$?

$$\operatorname{im} \phi = \{ax^2+bx+a+b \,;\, a,b,c \in \mathbb{R}\}$$

$$\operatorname{im} \phi \longleftrightarrow C(A_{\phi,\mathcal{B},\varphi})$$

$$C(A) = \left\{ \begin{array}{c} \begin{bmatrix} a+b \\ b \\ a \end{bmatrix} \end{array} \begin{array}{c} 1 \\ x \\ x^2 \end{array} ; \, a,b \in \mathbb{R} \right\}$$

Torej za $\tau : U \to V$ (lin. preslikava) velja:

① $\dim(\operatorname{im} \tau) = \operatorname{rang} A_{\tau,\mathcal{B},\varphi}$

② $\dim(\ker \tau) + \dim(\operatorname{im} \tau) = \dim U$