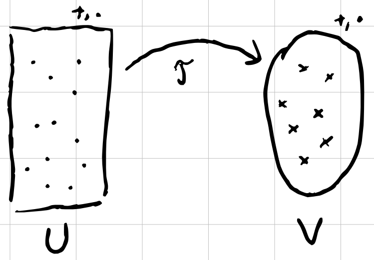
2.2. Linearne preslikave



Naj bosta U, V vektorska prostora.

Def: Preslikavo $\tau: U \to V$ imenujemo **linearna**, če:

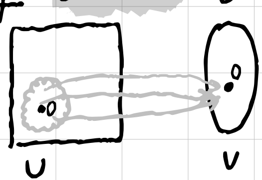
(1) $\tau(u_1 + u_2) = \tau(u_1) + \tau(u_2)$

(2) $\tau(\alpha u) = \alpha \tau(u)$

za vse $u, u_1, u_2 \in U$ in $\alpha \in \mathbb{R}$

Velja: Iz (2) sledi $\tau(0_u) = 0_v$. ničelni elementi

Def: **Jedro** $\tau: U \to V$ je $\ker(\tau) = \{u \in U; \tau(u) = 0_v\}$.



$\ker(\tau) \subseteq U$ je VPP v U.

$u_1, u_2 \in \ker(\tau) \Rightarrow \tau(u_1) = 0_v$ in $\tau(u_2) = 0_v$

Želimo pokazati $\alpha_1 u_1 + \alpha_2 u_2 \in \ker(\tau)$

$\tau(\alpha_1 u_1 + \alpha_2 u_2) \overset{(1)}{=} \tau(\alpha_1 u_1) + \tau(\alpha_2 u_2) \overset{(2)}{=}$

$= \alpha_1 \tau(u_1) + \alpha_2 \tau(u_2) = 0 \Rightarrow \alpha_1 u_1 + \alpha_2 u_2 \in \ker(\tau)$

Podobno kot zadnjič vidimo:

$\tau: U \to V$ je linearna natanko tedaj, ko
$$\tau(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \tau(u_1) + \alpha_2 \tau(u_2)$$
za $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ in $u_1, u_2 \in U$.

Primeri:

$\boxed{1}$ $\tau: \mathbb{R}^2 \to \mathbb{R}$

$\tau(\vec{u}) = \|\vec{u}\|_2$ <u>ni</u> linearna

$$\left.\begin{aligned} &\tau\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}3\\0\end{bmatrix}\right) = \tau\left(\begin{bmatrix}4\\1\end{bmatrix}\right) = \left\|\begin{bmatrix}4\\1\end{bmatrix}\right\| = \sqrt{17} \\ &\tau\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+\tau\left(\begin{bmatrix}3\\0\end{bmatrix}\right) = \sqrt{2}+3 \end{aligned}\right\} \Rightarrow \text{ni enako}$$

$\boxed{2}$ $\tau: U \to V$

$\tau(u) = 0$ je linearna

$\boxed{3}$ $A \in \mathbb{R}^{m \times n}$

$\tau: \mathbb{R}^n \to \mathbb{R}^m \quad \tau(\vec{x}) = A\vec{x}$ je linearna

Pokažimo. $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$

distributivnost
tsk. množenje

$$\tau(\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2) = A(\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2) = \alpha_1 A\vec{x}_1 + \alpha_2 A\vec{x}_2 =$$
$$= \alpha_1\tau(\vec{x}_1) + \alpha_2\tau(\vec{x}_2) \Rightarrow \tau \text{ je linearna}$$

$\ker(\tau) = \{\vec{x} \in \mathbb{R}^n ; A\vec{x} = 0\} = N(A)$ ničelni prostor

$\boxed{4}$ $\mathcal{D}: \mathbb{R}_n[x] \to \mathbb{R}_n[x]$

$\mathcal{D}(p) = p'$ Ali je $\mathcal{D}$ linearna?

$$\mathcal{D}(\alpha p + \beta q) = (\alpha p + \beta q)' = (\alpha p)' + (\beta q)' = \alpha p' + \beta q' =$$
$$= \alpha\mathcal{D}(p) + \beta\mathcal{D}(q)$$

$\Rightarrow \mathcal{D}$ je linearna preslikava

$\ker \mathcal{D} = \{p \in \mathbb{R}_n[x] ; p'(x) = 0 \text{ za } \forall x\}$
$= \{c ; c \in \mathbb{R}\}$ konstantni polinomi

Naj bo $B = \{b_1, \dots, b_m\}$ baza prostora U.
Potem lahko vsak $u \in U$ izrazimo kot
$$u = \alpha_1 b_1 + \dots + \alpha_m b_m$$
na en sam način, ker so $b_1, \dots, b_m$ lin. neodvisni.

Če poznamo $\tau(b_1), \tau(b_2), \dots, \tau(b_m)$, potem
$$\tau(u) = \tau(\alpha_1 b_1 + \dots + \alpha_m b_m) =$$
$$= \alpha_1 \tau(b_1) + \dots + \alpha_m \tau(b_m)$$
$\to$ Dovolj je poznati le slike baznih vektorjev B.

Vsak $\tau(b_i) \in V$. Naj bo $C = \{c_1, \dots, c_n\}$ baza V:
$$\tau(b_i) = \alpha_{1i} c_1 + \alpha_{2i} c_2 + \dots + \alpha_{ni} c_n$$
za $i = 1, \dots, m$.

Matriko, ki pripada linearni preslikavi τ iz baze B v bazo C:

$$A_{\tau,A,B} = \begin{array}{c c}
\overbrace{\begin{array}{cccc} b_1 & b_2 & \dots & b_m \\ \downarrow & \downarrow & & \downarrow \\ \tau(b_1) & \tau(b_2) & & \tau(b_m) \end{array}}^{m} & \\
\begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \alpha_{31} & \alpha_{32} & \dots & \alpha_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nm} \end{bmatrix} & \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{array}
\end{array} \in \mathbb{R}^{n \times m}$$

Koeficienti pri razvoju $\tau(b_1)$ po $c_1, c_2, \dots, c_n$.

Za neko bazo B in neko bazo C dobimo neko matriko $A_{\tau,B,C}$.

Primer:

[4] $\mathcal{D} : \mathbb{R}_n[x] \to \mathbb{R}_n[x] \qquad \mathcal{D}(p) = p'$

Zapiši matriko, ki pripada $\mathcal{D}$ iz standardne baze v standardno bazo $\mathbb{R}_n[x]$.

$\mathcal{S} = \{1, x, x^2, \dots, x^n\}$

$A_{\mathcal{D},\mathcal{S},\mathcal{S}} \in \mathbb{R}^{(n+1)\times(n+1)}$

$$A_{\mathcal{D},\mathcal{S},\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\boxed{b_1 = 1}$

$\mathcal{D}(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n$

$\boxed{b_2 = x}$

$\mathcal{D}(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n$

$\boxed{b_3 = x^2}$

$\mathcal{D}(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n$

$\boxed{b_n = x^n}$

$\mathcal{D}(x^n) = n x^{n-1} = 0 \cdot 1 + 0 \cdot x + \dots + n x^{n-1} + 0 \cdot x^n$

Razvoj po std. bazi

Naučili smo se: Če je mogoče, pišite matrike, ki ustrezajo lin. preslikavi iz * baze v STANDARDNO BAZO.

Še en primer:

$\Sigma: x+2y-3z=0$ (ravnina v $\mathbb{R}^3$)

Zapišite matriko, ki ustreza zrcaljenju Z čez Σ iz vaše najljubše baze za ta primer v poljubno bazo.

$B=\{\vec{n}, \vec{a}, \vec{b}\}$

$\vec{n}=\begin{bmatrix}1\\2\\-3\end{bmatrix}\in\mathbb{R}$, $\vec{a}=\begin{bmatrix}1\\1\\1\end{bmatrix}\in\Sigma$, $\vec{b}=\begin{bmatrix}3\\0\\1\end{bmatrix}\in\Sigma$, $\vec{a}\perp\vec{b}$

izbrali, ker ustreza enačbi Σ

$$\begin{matrix}\vec{n} & \vec{a} & \vec{b}\\ \downarrow & \downarrow & \downarrow\\ -\vec{n} & \vec{a} & \vec{b}\end{matrix}$$

$Z(\vec{n})=-\vec{n}=(-1)\vec{n}+0\cdot\quad+0\cdot$

$Z(\vec{a})=\vec{a}=0\cdot\vec{n}+1\cdot\vec{a}+0\cdot$

$Z(\vec{b})=\vec{b}=0\vec{n}+0\vec{a}+1\cdot\vec{b}$

$$A_{Z,B,B}=\begin{array}{c}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{array}\begin{array}{c}\vec{n}\\\vec{a}\\\vec{b}\end{array}$$

Kaj pa matrika, ki pripada Z iz std. baze $\mathbb{R}^3$ v std. bazo $\mathbb{R}^3$?

$B=\{\vec{n},\vec{a},\vec{b}\}$ $\quad S=\{\vec{i},\vec{j},\vec{k}\}$

Imamo $A_{Z,B,B}$, želimo $A_{Z,S,S}$.

$$\begin{tikzcd}
(\mathbb{R}^3, B) \arrow[r, bend left, "A_{Z,B,B}"] & (\mathbb{R}^3, B) \arrow[d, "P"] \\
{[\mathbb{R}^3, S]} \arrow[u, "Q=P^{-1}"] \arrow[r, "A_{Z,S,S}\,?"] & {[\mathbb{R}^3, S]}
\end{tikzcd}$$

$$\begin{array}{c}\begin{array}{ccc}\vec{n}&\vec{a}&\vec{b}\end{array}\\ P=\begin{bmatrix}1&1&3\\2&1&0\\-3&1&1\end{bmatrix}\end{array}\begin{array}{c}\vec{i}\\\vec{j}\\\vec{k}\end{array}$$

matrika, ki bazo B razvije po bazi S

Verjamemo:

1. Kompozitum linearnih preslikav je linearna preslikava. Matrika, ki ustreza kompozitumu je produkt pripadajočih matrik.

$$\begin{tikzcd} U \arrow[r, "\tau"] \arrow[rr, bend left, "\psi\circ\tau"] & V \arrow[r, "\psi"] & W \\ \mathcal{B} & \mathcal{C} & \mathcal{G} \end{tikzcd}$$

2. Če τ obrnljiva ($\Leftrightarrow \tau$ bijekcija), potem $A_{\tau^{-1},\mathcal{C},\mathcal{B}} = (A_{\tau,\mathcal{B},\mathcal{C}})^{-1}$

$$\begin{tikzcd} V \arrow[r, bend left, "\tau"] & U \arrow[l, bend left, "\tau^{-1}"] \\ \mathcal{B} & \mathcal{C} \end{tikzcd}$$

$$\boxed{A_{\tau,\mathcal{G},\mathcal{G}} = P\cdot A_{\tau,\mathcal{B},\mathcal{B}}\cdot P^{-1}}$$

Matrike, ki ustrezajo dani lin. preslikavi v različnih bazah, so si med seboj podobne.

<u>Def</u>: Lastne vrednosti $\tau: U\to U$ so lastne vrednosti matrike $A_{\tau,\mathcal{B},\mathcal{B}}$, ki pripada τ iz baze $\mathcal{B}$ v bazo $\mathcal{B}$.

Če je λ lastna vrednost τ, obstaja tak vektor $u\neq 0$, $u\in U$, da velja $\tau(u)=\lambda u$.

Primer:

a) $\mathcal{Z}: \mathbb{R}^3 \to \mathbb{R}^3$, zrcaljenje čez $x+2y-3z=0$.

Kaj so lastne vrednosti?

Ker $A_{\mathcal{Z},B,B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so l. vrednosti $\mathcal{Z}$ enake $-1, 1, 1$.

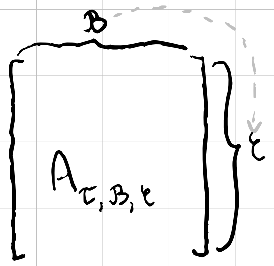
Pripadajoči l. vektorji: $\vec{n}, \vec{a}, b$

b) Zapišimo matriko, ki pripada $\mathcal{Z}$ iz S v S.

$$A_{\mathcal{Z},S,S} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 6 & -2 & 3 \\ -2 & 3 & 6 \\ 3 & 6 & -2 \end{bmatrix}$$

c) Kam se zrcali vektor $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$?

$$\mathcal{Z}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = A_{\mathcal{Z},S,S} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$



$$\tau(\alpha_1 b_1 + \ldots + \alpha_m b_m) =$$
$$= \alpha_1 \tau(b_1) + \ldots + \alpha_m \tau(b_m) =$$
$$= \alpha_1(\alpha_{11} c_1 + \ldots + \alpha_{n1} c_n) + \ldots + \alpha_m(\alpha_{1m} c_1 + \ldots + \alpha_{nm} c_n)$$
$$= (\alpha_1 \alpha_{11} + \ldots + \alpha_m \alpha_{1m}) c_1 + \ldots + (\_\_\_) c_n$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$$

$u \rightsquigarrow$ razvijanje po B

$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$ ← vektor koeficientov razvoja u po B

$\tau(u)$ $\quad A_{\tau,B,C} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$ ← koef. $\tau(u)$ pri razvoju po bazi C