

1. Računski izpit iz Matematike 1

31. januar 2025

Čas pisanja: **90 minut**. Dovoljena je uporaba dveh listov velikosti A4 za pomoč. Prepisovanje, pogovarjanje in uporaba knjig, zapiskov, pametnega telefona in ostalih elektronskih naprav je **strogo prepovedano**.

1. naloga (25 točk)Naj bo A matrika

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

a) (5 točk) Ali je matrika A pozitivno definitna? Ali obstaja razcep Choleskega za matriko A ? Če obstaja, ga poišči.

b) (20 točk) Poišči pozitivno semidefinitno, vendar ne strogo definitno, matriko A' , ki je v Frobeniusovi normi najbližja matriki A .

(a) Uporabimo Sylvestrov kriterij:

$1 > 0$, $\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8 < 0 \dots$ A ni pozitivno definitna in
zato nima razcepa Choleskega.

(b) Poiščimo singularne vrednosti A oz. kar lastne vrednosti A (saj je A simetrična).

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 1-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 3 & -1 \\ 2+\lambda & 1-\lambda & -1 \\ 0 & -1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 3 & -1 \\ 0 & 4-\lambda & -2 \\ 0 & -1 & 5-\lambda \end{vmatrix} =$$

$$= (-2-\lambda)(\lambda^2 - 9\lambda + 18) = (-2-\lambda)(\lambda-3)(\lambda-6) = 0$$

$$\text{lastne vrednosti} \rightarrow \lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$$

A' bo ranga 2, največji singularni vrednosti A sta $\sigma_2 = \lambda_2 = 3$ in $\sigma_3 = \lambda_3 = 6$. Poiščimo pripadajoče singularne/lastne vektorje:

$$\bullet \lambda_2 = 3: A - 3I = \begin{bmatrix} -2 & 3 & -1 \\ 3 & -2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} +1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} x - z = 0 \\ y - z = 0 \\ \gamma - z = 0 \end{matrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \dots \vec{\hat{z}}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\bullet \lambda_3 = 6: A - 6I = \begin{bmatrix} -5 & 3 & -1 \\ 3 & -5 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} x + \frac{1}{2}z = 0 \\ y + \frac{1}{2}z = 0 \\ \gamma + \frac{1}{2}z = 0 \end{matrix} \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{\hat{z}}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Po Eckart-Youngovem izreku je:

$$A' = 3\vec{\hat{z}}_2\vec{\hat{z}}_2^T + 6\vec{\hat{z}}_3\vec{\hat{z}}_3^T = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

2. naloga (25 točk)

Naj bo $\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ standardna baza $\mathbb{R}^2$. Preslikava $\phi: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}_2[x]$ je dana s predpisom

$$(\phi(A))(x) = (e_1^T A e_1)x^2 + (e_2^T A e_2)x + (e_1 + e_2)^T A (e_1 + e_2).$$

a) (15 točk) Prepričaj se, da je ϕ linearna preslikava in poišči matriko, ki pripada ϕ glede na standardni bazi prostorov $\mathbb{R}^{2 \times 2}$ in $\mathbb{R}_2[x]$.

b) (10 točk) Poišči bazi za jedro in sliko preslikave ϕ .

(a) Zapišemo lahko

$$(\phi(A))(x) = (x^2 \vec{e}_1 + x \vec{e}_2 + \vec{e}_1 + \vec{e}_2)^T A \underbrace{(x^2 \vec{e}_1 + x \vec{e}_2 + \vec{e}_1 + \vec{e}_2)}_{\vec{a}} = \vec{a}^T A \vec{a}.$$

$$\begin{aligned} \text{Velja: } (\phi(\alpha A + \beta B))(x) &= \vec{a}^T (\alpha A + \beta B) \vec{a} = \alpha \vec{a}^T A \vec{a} + \beta \vec{a}^T B \vec{a} = \\ &= \alpha (\phi(A))(x) + \beta (\phi(B))(x), \end{aligned}$$

tj. ϕ je linearna.

Poračunamo:

$$\phi(E_{11}) = x^2 + 1,$$

$$\phi(E_{12}) = 1,$$

$$\phi(E_{21}) = 1,$$

$$\phi(E_{22}) = x + 1.$$

Torej

$$A_\phi = \begin{bmatrix} E_{11} & E_{12} & E_{21} & E_{22} \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ 0 \end{bmatrix}$$

(b) Ker je $\text{rang } A_\phi = 3$, je $\dim(\text{im } \phi) = 3$, torej $\text{im } \phi = \mathbb{R}_2[x]$ in zato $B_{\text{im } \phi} = \{1, x, x^2\}$.

$$A_\phi \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ tj. v } N(A_\phi) \text{ so vektorji oblike } \begin{bmatrix} 0 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix},$$

kar pomeni $-x_3 E_{12} + x_3 E_{21} \in \ker \phi$. $B_{\ker \phi} = \{-E_{12} + E_{21}\}.$

3. naloga (25 točk)

Izračunaj prostornino območja $D \subset \mathbb{R}^3$ danega z neenačbami

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 2, z \geq x^2 + y^2\}.$$

V valjnih koordinatah $\begin{pmatrix} x = r \cos \varphi \\ y = r \sin \varphi \\ z = z \end{pmatrix}$ dobimo neenačbost:

$$r^2 + z^2 \leq 2 \quad \text{in} \quad z \geq r^2 \quad \dots \quad r^2 \leq z \leq \sqrt{2-r^2}.$$

Meje za integracijo po z .

$$\text{Iz } r^2 \leq \sqrt{2-r^2} \text{ dobimo se } r^4 \leq 2-r^2 \dots r^4 + r^2 - 2 \leq 0 \dots$$

$$\dots r^2 = \frac{-1 \pm 3}{2} \dots r^2 = 1 \dots r = 1.$$

Ker v neenačbostih ni pogoja za φ , je $0 \leq \varphi \leq 2\pi$.

↑
Zgornja meja
za r .

Dobimo:

$$V = \int_0^{2\pi} \left(\int_0^1 \left(\int_{r^2}^{\sqrt{2-r^2}} r \cdot dz \right) dr \right) d\varphi = 2\pi \int_0^1 r (\sqrt{2-r^2} - r^2) dr =$$

$\det(J_{\vec{F}})$

$$= 2\pi \left(\underbrace{\int_0^1 r \sqrt{2-r^2} dr}_{\substack{t=2-r^2 \\ dt = -2r dr \dots r dr = -\frac{dt}{2}}} - \underbrace{\int_0^1 r^3 dr}_{\substack{r^4 \\ 4 \big|_{r=0}^{r=1} = \frac{1}{4}}} \right) = \pi \left(\frac{4}{3} \sqrt{2} - \frac{7}{6} \right).$$

$$= -\frac{1}{2} \int_2^1 \sqrt{t} dt = \frac{1}{2} \frac{t^{3/2}}{3/2} \bigg|_{t=1}^{t=2} = \frac{1}{3} (2\sqrt{2} - 1)$$

4. naloga (25 točk)

Naj bosta dana vektorja $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ ter pozitivno definitna matrika $A \in \mathbb{R}^{n \times n}$. Izrazi največjo in najmanjšo vrednost funkcije

$$f(\mathbf{x}) = \mathbf{a}^T \mathbf{x}$$

pri pogojih

$$\mathbf{x}^T A^{-1} \mathbf{x} = 1 \text{ in } \mathbf{b}^T \mathbf{x} = 0.$$

Pripadajoča Lagrangeova funkcija je:

$$L(\bar{\mathbf{x}}, \lambda, \mu) = \bar{\mathbf{a}}^T \bar{\mathbf{x}} - \lambda (\bar{\mathbf{x}}^T A^{-1} \bar{\mathbf{x}} - 1) - \mu \mathbf{b}^T \bar{\mathbf{x}}.$$

saj je A^{-1} tudi simetrična

$$\frac{\partial L}{\partial \bar{\mathbf{x}}} = \begin{cases} \bar{\mathbf{a}}^T - 2\lambda \bar{\mathbf{x}}^T A^{-1} - \mu \mathbf{b}^T = \vec{0}^T \\ \bar{\mathbf{x}}^T A^{-1} \bar{\mathbf{x}} = 1 \\ \mathbf{b}^T \bar{\mathbf{x}} = 0 \end{cases} \quad \left. \vphantom{\frac{\partial L}{\partial \bar{\mathbf{x}}}} \right\} \text{rešiti moramo ta sistem.}$$

$$2\lambda A^{-1} \bar{\mathbf{x}} = \bar{\mathbf{a}} - \mu \mathbf{b} \quad \dots \quad \bar{\mathbf{x}} = \frac{1}{2\lambda} A(\bar{\mathbf{a}} - \mu \mathbf{b}) \quad \dots \quad \mathbf{b}^T \bar{\mathbf{x}} = \frac{1}{2\lambda} \mathbf{b}^T A(\bar{\mathbf{a}} - \mu \mathbf{b}) = 0$$

če $\lambda \neq 0$

$$\dots \mathbf{b}^T A \bar{\mathbf{a}} - \mu \mathbf{b}^T A \mathbf{b} = 0$$

$$\dots \mu = \frac{\mathbf{b}^T A \bar{\mathbf{a}}}{\mathbf{b}^T A \mathbf{b}}$$

$$\dots (\bar{\mathbf{a}} - \mu \mathbf{b})^T A (\bar{\mathbf{a}} - \mu \mathbf{b}) = 4\lambda^2 \dots$$

$$\dots \bar{\mathbf{a}}^T A \bar{\mathbf{a}} - 2\mu \mathbf{b}^T A \bar{\mathbf{a}} + \mu^2 \mathbf{b}^T A \mathbf{b} = 4\lambda^2$$

ker je A simetrična

$$\dots \bar{\mathbf{a}}^T A \bar{\mathbf{a}} - 2 \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}} + \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}} = 4\lambda^2$$

$$\dots \lambda = \pm \frac{1}{2} \sqrt{\bar{\mathbf{a}}^T A \bar{\mathbf{a}} - \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}}}$$

$$f(\bar{\mathbf{x}}) = \bar{\mathbf{a}}^T \bar{\mathbf{x}} = \pm \frac{\bar{\mathbf{a}}^T A \bar{\mathbf{a}} - \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}}}{\sqrt{\bar{\mathbf{a}}^T A \bar{\mathbf{a}} - \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}}}} = \pm \sqrt{\bar{\mathbf{a}}^T A \bar{\mathbf{a}} - \frac{(\mathbf{b}^T A \bar{\mathbf{a}})^2}{\mathbf{b}^T A \mathbf{b}}} \leftarrow \text{največja in najmanjša vrednost.}$$