Name and surname: ______________________ Student ID: ____________________

EXAM in MATHEMATICS 1, THEORETICAL PART
January 27, 2026

All tasks require full justification unless stated otherwise.

1. **(10 points)** Prove that $\|\vec{x}\,\vec{y}^{\mathsf{T}}\|_F = \|\vec{x}\|\,\|\vec{y}\|$ holds for arbitrary vectors $\vec{x} \in \mathbb{R}^m$ and $\vec{y} \in \mathbb{R}^n$.

$$\|\vec x \vec y^T\|_F^2 = \langle \vec x \vec y^T, \vec x \vec y^T \rangle_F = tr(\vec x \vec y^T (\vec x \vec y^T)^T) = tr(\vec x \underbrace{\vec y^T \vec y}_{=\|\vec y\|^2 \in \mathbb{R}} \vec x^T) =$$

$$= \|\vec y\|^2 \, tr(\vec x \vec x^T) = \|\vec y\|^2 \, tr(\underbrace{\vec x^T \vec x}_{=\|\vec x\|^2 \in \mathbb{R}}) \Rightarrow$$

$$\underbrace{\|\vec x \vec y^T\|_F^2}_{\geq 0} = \underbrace{\|\vec y\|^2}_{\geq 0} \underbrace{\|\vec x\|^2}_{\geq 0} \quad /\sqrt{.}$$

$$\Rightarrow \|\vec x \vec y^T\|_F = \|\vec x\| \, \|\vec y\|$$

2. **([redacted] points)** Write an example of a vector subspace of $\mathbb{R}^{3\times 3}$ of dimension 4 that contains a matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

No justification is required.

E.g.

$$V = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} ; a, b, c, d \in \mathbb{R} \right\}$$

$$\text{or } = \mathcal{L}\{ I, E_{12}, E_{13}, E_{23} \}$$

3. **(10 points)** Write an example of nonsymmetric matrices $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{3\times 3}$, such that $A \otimes B$ has the Frobenius norm at most $\sqrt{5}$ and exactly two distinct eigenvalues, one with multiplicity 2 and one with multiplicity 4. No justification is required.

any "small" values

$$A = \begin{bmatrix} \boxed{1/10} & \boxed{1/10} \\ \boxed{0} & \boxed{1/10} \end{bmatrix} \qquad B = \begin{bmatrix} \boxed{0} & \boxed{0} & \boxed{1/10} \\ \boxed{0} & \boxed{1/10} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{1/10} \end{bmatrix}$$

$$\|A \otimes B\|_F^2 = \|A\|_F^2 \, \|B\|_F^2 = \frac{3}{100} \cdot \frac{3}{100} = \frac{9}{10^4} < 5$$

4. **(10 points)** Let τ, ψ[redacted]: $V \to V$ be linear transformations of an n-dimensional vector space V.

A. Suppose $\tau^2 = \tau \circ \tau = 0$. Show that $\mathrm{im}(\tau) \subseteq \ker(\tau)$.

5

$u \in \mathrm{im}(\tau) \Rightarrow \exists v \in V : \tau(v) = u$
$\Rightarrow \tau(u) = \tau(\tau(v)) = \tau^2(v) = 0_V$
$\Rightarrow u \in \ker(\tau)$
$\Rightarrow \mathrm{im}(\tau) \subseteq \ker(\tau)$

B. In case $V = \mathbb{R}_4[x]$, write an example of ψ which satisfies $\dim(\ker(\psi)) \leq 3$ and $\psi^2 = 0$. Write an explicit formula and prove that it satisfies the two desired properties.

5

$\psi : \mathbb{R}_4[x] \to \mathbb{R}_4[x]$
e.g. $\psi(a_4x^4 + a_3x^3 + \dots + a_0) = a_4x^2 + a_3$
• $\psi^2(a_4x^4 + \dots + a_0) = \psi(a_4x^2 + a_3) = 0$ ✓
• $a_4x^4 + \dots + a_0 \in \ker\psi \Rightarrow a_4x^2 + a_3 = 0 \ \forall x \Rightarrow a_4 = a_3 = 0$
$\Rightarrow \ker\psi \subseteq \mathrm{Lin}\{1, x, x^2\} \Rightarrow \dim(\ker\psi) \leq 3$ ✓

(e.g. $\psi(p) = p''$, ...)

5. **(15 points)** Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let matrix $B \in \mathbb{R}^{n\times n}$ be invertible. Show that matrix $B^\top AB$ is symmetric positive definite as well.

$\underbrace{A = A^T}_{(*)}$, $\underbrace{A \text{ PD}}_{(*)}$, $\underbrace{B \text{ invertible}}_{(*)}$

5

• B^TAB symm: $(B^TAB)^T = B^TA^TB \overset{A \text{ symm.}}{=} B^TAB \Rightarrow$ symmetric

• B^TAB PD $\quad \vec{x}^T(B^TAB)\vec{x} = (\vec{x}^TB^T)A(B\vec{x}) =$
$= (B\vec{x})^TA(B\vec{x}) \quad \vec{y} := B\vec{x}$ 5
$= \vec{y}^TA\vec{y} \underset{A \text{ PD}}{\geq 0}$

$\vec{x}^T(B^TAB)\vec{x} = 0 \Leftrightarrow \vec{y}^TA\vec{y} = 0$ for $\vec{y} = B\vec{x}$
$\Leftrightarrow \vec{y} = \vec{0}$ since A PD 5
$\Leftrightarrow B\vec{x} = \vec{0}$
$\Leftrightarrow \vec{x} = \vec{0}$ since B invertible

6. **(15 points)** For arbitrary $b, c \in \mathbb{R}$ set $M = \begin{bmatrix} 4 & b & 0 \\ b & c & 0 \\ 0 & 0 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 3}$.

A. Write an example of a function f, a point $\mathbf{a} \in \mathcal{D}_f$ and nonzero $b, c \in \mathbb{R}$, such that $H_f(\mathbf{a}) = M$.

e.g. $f(x,y,z) = 2x^2 + xy + y^2 + z^2$
$\underline{a} = (0,0,0)$
$b = 1$
$c = 2$

5

$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 4$
$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 1$
$\frac{\partial^2 f}{\partial y^2}(x,y,z) = 2$ $\quad \Rightarrow H_f(\underline{a}) = M$
$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0$
$\frac{\partial^2 f}{\partial z^2} = 2$

B. Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable function and let $\mathbf{a} \in \mathcal{D}_g$ be a stationary/critical point of g, such that $H_g(\mathbf{a}) = M$.

B1. In case $b \neq 0$ and $c = 0$, can $\mathbf{a}$ be a local *maximum* of g? If yes, write an example of such an $H_g(\mathbf{a})$. If not, then explain why not.

5

$$M = \begin{bmatrix} 4 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- one of eigvals of M is $2 > 0$
- $\det(M) = 2(0 - b^2) = -2b^2 < 0$

$\Rightarrow$ one of eigenvals of M is negative

$\Rightarrow$ M is indefinite

$\Rightarrow$ g cannot have local extremum.

$\Rightarrow$ $\underline{a}$ is not a local max

(You got 2,5 points if you proved M cannot be ND. Recall that local max can happen even if $\det M = 0$.)

B2. In case $c \neq 0$, can $\mathbf{a}$ be a local *minimum* of g? If yes, write an example of such an $H_g(\mathbf{a})$. If not, then explain why not.

5

YES.

e.g. $H_g(\underline{a}) = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad + \; + \; +$

Sylvester $\Rightarrow$ $H_g(\underline{a})$ is PD

$\Rightarrow$ $\underline{a}$ is local minimum

7. **(15 points)** Let

$$\mathcal{D} = \{(x, y, z) : x \leq 0,\ y \geq |x|,\ 0 \leq z \leq x^2 + y^2 \leq 4\} \subseteq \mathbb{R}^3.$$

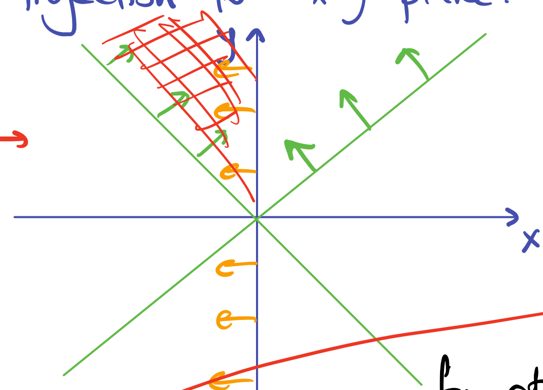
Sketch the projection of $\mathcal{D}$ on x-y plane ($z = 0$), and express the integral

$$I = \iiint_{\mathcal{D}} \underbrace{(x^2 + y^2 + z^2)}_{r^2 + z^2}\, dx\, dy\, dz$$

in cylindrical coordinates. (Do not evaluate the integral.)

Projection to x-y plane:

5



$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$\varphi \in [0, 2\pi]$

$x \leq 0$ & $y \geq |x|$: $\quad \varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

$0 \leq z \leq x^2 + y^2 \leq 4$: $\quad 0 \leq z \leq r^2 \leq 4$

5

function 2,5

det of Jacobian 2,5

$$I = \int_{\pi/2}^{3\pi/4} \left(\int_0^2 \left(\left(\int_0^{r^2} (r^2 + z^2)\, r \right) dz \right) dr \right) d\varphi$$

(or

$$I = \int_{\pi/2}^{3\pi/4} \left(\int_0^4 \left(\left(\int_{\sqrt{z}}^{2} (r^2 + z^2)\, r \right) dr \right) dz \right) d\varphi$$

)

8. **(17 points)** Let A be a symmetric positive definite $n \times n$ matrix and let $\vec{b}, \vec{c} \in \mathbb{R}^n$. We wish to find the minimum value of the function $f(\vec{x}) = \vec{x}^{\top} A \vec{x} - \vec{b}^{\top} \vec{x}$ subject to the constraints $\|\vec{x}\|^2 \leq 1$ and $\vec{c}^{\top} \vec{x} = 4$.

$\|\vec{x}\|^2 - 1 \leq 0 \leftarrow \mu \leq 0$

$\vec{c}^{T}\vec{x} - 4 = 0 \leftarrow \lambda$

3 A. Write down the Lagrangian function for this problem.

$$L(\vec{x}, \lambda, \mu) = \vec{x}^T A \vec{x} - \vec{b}^T \vec{x} - \lambda(\vec{c}^T \vec{x} - 4) - \mu(\|x\|^2 - 1)$$

1 1 1

6 B. State the Karush–Kuhn–Tucker conditions for this problem.

$$0 = \frac{\partial L}{\partial \vec{x}} = 2\vec{x}^T A - \vec{b}^T - \lambda \vec{c}^T - 2\mu \vec{x}^T$$

$$\xrightarrow{T} \boxed{2A\vec{x} - \vec{b} - \lambda \vec{c} - 2\mu \vec{x} = \vec{0}} \quad 2$$

$\boxed{\|\vec{x}\|^2 - 1 \leq 0}$ 1

$\boxed{\mu \leq 0}$ 1

$\boxed{\mu(\|x\|^2 - 1) = 0}$ 1

$\boxed{\vec{c}^T \vec{x} - 4 = 0}$ 1

8 C. Show that the solution of Karush-Kuhn-Tucker conditions is the same as the solution to the mentioned problem. (Do not solve either of the two problems.)

1 This follows from Slater's conditions:

3 1) $f(\vec{x}) = \vec{x}^T A \vec{x} - \vec{b}^T \vec{x}$ is convex:

$$\frac{\partial f}{\partial \vec{x}} = 2\vec{x}^T A - \vec{b}^T$$

$$\frac{\partial^2 f}{\partial \vec{x}^2} = \frac{\partial}{\partial x}\left(2A\vec{x} - \vec{b}^T\right) = 2A$$

since A PD $\Rightarrow 2A$ is PD $\Rightarrow f$ convex ✓

3 2) $h(\vec{x}) = \|\vec{x}\|^2 - 1$ is convex:

$$\frac{\partial h}{\partial \vec{x}} = 2\vec{x}^T \Rightarrow \frac{\partial^2 h}{\partial \vec{x}^2} = \frac{\partial (2\vec{x})}{\partial \vec{x}} = 2I \text{ is PD}$$

$\Rightarrow h$ convex ✓

1 3) $g(\vec{x}) = \vec{c}^T \vec{x} - 4$ is affine ✓