Ime in priimek

Vpisna številka

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**1. Računski izpit iz Matematike 1**

27. januar 2026

Čas pisanja: **90 minut**. *Dovoljena je uporaba dveh listov velikosti A4 za pomoč. Prepisovanje, pogovarjanje in uporaba knjig, zapiskov, pametnega telefona in ostalih elektronskih naprav je* **strogo** *prepovedano.*

**1. naloga (25 točk)**

Naj bo $\mathbf{a} = [1, -1]^\mathsf{T} \in \mathbb{R}^2$. Preslikava $\psi\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ ima predpis

$$\psi(X) = (\mathbf{a} \otimes X + X \otimes \mathbf{a})\mathbf{a}.$$

**a) (5 točk)** Prepričaj se, da je $\psi$ linearna preslikava.

**b) (10 točk)** Poišči matriko, ki pripada $\psi$ glede na standardni bazi $\mathbb{R}^{2\times 2}$ in $\mathbb{R}^4$.

**c) (10 točk)** Poišči bazi za $\ker\psi$ in $\operatorname{im}\psi$.

(a) Za $X, Y \in \mathbb{R}^{2\times 2}$ in $\alpha, \beta \in \mathbb{R}$ velja:

$$\psi(\alpha X + \beta Y) = (\vec{a} \otimes (\alpha X + \beta Y) + (\alpha X + \beta Y) \otimes \vec{a})\vec{a} =$$
$$= (\alpha \vec{a} \otimes X + \beta \vec{a} \otimes Y + \alpha X \otimes \vec{a} + \beta Y \otimes \vec{a})\vec{a} =$$
$$= \alpha(\vec{a} \otimes X + X \otimes \vec{a})\vec{a} + \beta(\vec{a} \otimes Y + Y \otimes \vec{a})\vec{a} =$$
$$= \alpha\psi(X) + \beta\psi(Y),$$ tj. $\psi$ je linearna.

(b) $S_{\mathbb{R}^{2\times2}} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $S_{\mathbb{R}^4} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$, $\vec{a} = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\psi(E_{11}) = \Bigg(\underbrace{\begin{bmatrix}1&0\\0&0\\-1&0\\0&0\end{bmatrix}}_{=\vec{a}\otimes E_{11}} + \underbrace{\begin{bmatrix}1&0\\-1&0\\0&0\\0&0\end{bmatrix}}_{=E_{11}\otimes\vec{a}}\Bigg)\underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{=\vec{a}} = \begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}, \quad \psi(E_{12}) = \left(\begin{bmatrix}0&1\\0&0\\0&-1\\0&0\end{bmatrix} + \begin{bmatrix}0&1\\0&-1\\0&0\\0&0\end{bmatrix}\right)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\\0\end{bmatrix}$$

$$\psi(E_{21}) = \ldots = \begin{bmatrix}0\\-1\\-1\\2\end{bmatrix}, \quad \psi(E_{22}) = \ldots = \begin{bmatrix}0\\1\\1\\-2\end{bmatrix}$$ Torej $$A_\psi = \begin{bmatrix}2&-2&0&0\\-1&1&-1&1\\-1&1&-1&1\\0&0&2&-2\end{bmatrix}.$$

(c) $$A_\psi \to \begin{bmatrix}1&-1&0&0\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix},$$ torej $B_{C(A_\psi)} = \left\{\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\\2\end{bmatrix}\right\}$ in $B_{N(A_\psi)} = \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$

v bazi $S_{\mathbb{R}^4}$ $\updownarrow$ v bazi $S_{\mathbb{R}^{2\times2}}$ $\updownarrow$

$$B_{\operatorname{im}\psi} = \left\{\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\\2\end{bmatrix}\right\} \qquad B_{\ker\psi} = \left\{\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$$

**2. naloga (25 točk)**

Naj bo $A \in \mathbb{R}^{2\times 2}$ matrika

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

**a) (10 točk)** Brez računanja matrike $A\otimes A$ poišči lastne vrednosti in pripadajoče lastne vektorje matrike $A \otimes A$.

**b) (5 točk)** Ali ima $A \otimes A$ razcep Choleskega? Če ga ima, ga poišči.

**c) (10 točk)** Poišči matriko $A_1$ ranga 1, ki minimizira vrednost $\|A \otimes A - A_1\|_F$.

(a) Lastni vrednosti $A$ sta $\lambda_1 = 6$ in $\lambda_2 = 2$, z lastnima vektorjema $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(Upoštevamo, da je vsota obeh vrstic enaka, in, da je $A$ simetrična.)

Lastne vrednosti $A\otimes A$ so torej $\lambda_1^2 = 36$, $\lambda_1 \cdot \lambda_2 = 12$, $\lambda_1 \cdot \lambda_2 = 12$, $\lambda_2^2 = 4$

z lastnimi vektorji $\vec{v}_1 \otimes \vec{v}_1$, $\vec{v}_1 \otimes \vec{v}_2$, $\vec{v}_2 \otimes \vec{v}_1$, $\vec{v}_2 \otimes \vec{v}_2$.

(b) Ker je $A\otimes A$ pozitivno definitna in simetrična, ima razcep Choleskega.

Poiščimo razcep Choleskega za $A$:

$$L_1 = \begin{bmatrix} \sqrt{4} & 0 \\ \frac{2}{\sqrt{4}} & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_2 = 4 - \frac{1}{4} \cdot 2 \cdot 2 = 3 \quad \dots \quad L_2 = \sqrt{3}$$

$$L = \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}.$$

V razcepu Choleskega za $A\otimes A$ je sp. trikotni faktor torej $L\otimes L$.

(c) Normirajmo lastni vektor $\vec{v}_1 \otimes \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ za $A\otimes A$ ... $\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Po Eckart-Youngovem izreku je:

$$A_1 = \lambda_1^2 \, \vec{q}_1 \vec{q}_1^T = 9 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

**3. naloga (25 točk)**

Telo $T \subseteq \mathbb{R}^3$ je določeno z neenačbami

$$z^2 \geq x^2 + y^2\ ,\ z \leq 6 - x^2 - y^2 \ \text{ in } \ x + y \geq 0.$$

Izračunaj prostornino in maso tega telesa, če je gostota enaka $\rho(x,y,z) = x^2$.

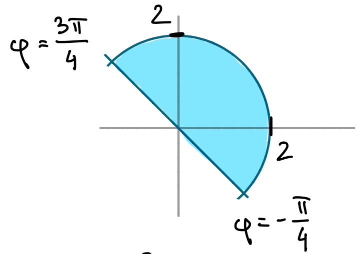
V valjnih koordinatah $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$ je telo opisano z: $z^2 \geq r^2 \ldots z \geq r$

$z \leq 6 - r^2$

in $r(\cos\varphi + \sin\varphi) \geq 0 \ldots$

$\ldots\ \varphi \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Skica projekcije $T$ na $xy$-ravnino:




Iz $r \leq z \leq 6 - r^2$

dobimo $r \leq 6 - r^2 \ \ldots\ r^2 + r - 6 \leq 0$

$\ldots\ (r+3)(r-2) \leq 0 \ \ldots\ r \in [-3, 2]$ oz.

$r \in [0, 2]$ (saj $r \geq 0$).

$$V = \iiint_T dx\,dy\,dz = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left( \int_0^2 \left( \int_r^{6-r^2} \underset{\substack{\uparrow \\ \det J_{\vec{F}}}}{r} \cdot dz \right) dr \right) d\varphi = \underbrace{\left( \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \right)}_{=\pi} \left( \int_0^2 r\,(6 - r^2 - r)\, dr \right) =$$

$$= \pi \left( 3r^2 - \frac{r^4}{4} - \frac{r^3}{3} \right) \Bigg|_{r=0}^{r=2} = \frac{16\pi}{3} \leftarrow \text{prostornina}$$

$$m = \iiint_T \rho\, dx\,dy\,dz = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left( \int_0^2 \left( \int_r^{6-r^2} \underbrace{r^2\cos^2\varphi}_{\rho} \cdot \underset{\substack{\uparrow \\ \det J_{\vec{F}}}}{r}\, dz \right) dr \right) d\varphi =$$

$$= \underbrace{\left( \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2\varphi\, d\varphi \right)}_{\pi \cdot \frac{1}{2}} \underbrace{\left( \int_0^2 r^3 (6 - r^2 - r)\, dr \right)}_{\left( \frac{3r^4}{2} - \frac{r^6}{6} - \frac{r^5}{5} \right) \Big|_{r=0}^{r=2} = 24 - \frac{32}{3} - \frac{32}{5}} = \frac{52\pi}{15} \leftarrow \text{masa}$$

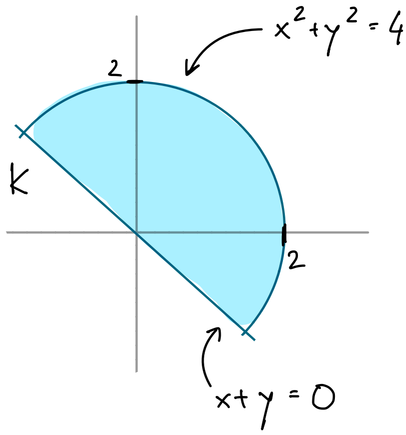
**4. naloga (25 točk)**

Poišči največjo in najmanjšo vrednost funkcije

$$f(x,y) = x^2 + xy + y^2$$

na podmnožici $K \subseteq \mathbb{R}^2$ določeni z neenačbama

$$x^2 + y^2 \leq 4 \ \text{ in } \ x + y \geq 0.$$




f lahko največjo/najmanjšo vrednost zavzame v notranjosti ali na robu K.

- notranjost K, določena z neenačbama $x^2+y^2<4$ in $x+y>0$:

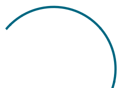
$$\frac{\partial f}{\partial x} = 2x+y = 0 \qquad \frac{\partial f}{\partial y} = x+2y = 0$$

Ta sistem ima edino rešitev $(x,y)=(0,0)$, ki pa ni v notranjosti K.

- rob K, sestavljen iz

$x+y=0$, $x^2+y^2 \leq 4$

in



$x^2+y^2=4$, $x+y \geq 0$

$$L_1(x,y,\lambda) = \underbrace{x^2+xy+y^2}_{f(x,y)} - \lambda(x+y) \ \ldots$$

$$\left.\begin{aligned} \frac{\partial L_1}{\partial x} &= 2x+y-\lambda = 0 \\ \frac{\partial L_1}{\partial y} &= x+2y-\lambda = 0 \end{aligned}\right\} \quad 2x+y = x+2y, \ \text{tj. } x-y=0$$

Skupaj z vezjo $x+y=0$ dobimo $(x,y)=(0,0)$.

$$L_2(x,y,\lambda) = x^2+xy+y^2-\lambda(x^2+y^2-4)$$

$$\left.\begin{aligned} \frac{\partial L_2}{\partial x} &= 2x+y-2\lambda x = 0 \ \ldots \ 2x+y = 2\lambda x \\ \frac{\partial L_2}{\partial y} &= x+2y-2\lambda y = 0 \ \ldots \ x+2y = 2\lambda y \end{aligned}\right\} \ \ldots \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrika $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ ima lastni vrednosti $2\lambda_1 = 3$ in $2\lambda_2 = 1$, pripadajoča l. vektorja sta $\vec{v}_1 = \begin{bmatrix} x \\ x \end{bmatrix}$ in $\vec{v}_2 = \begin{bmatrix} x \\ -x \end{bmatrix}$. Ko upoštevamo vez $x^2+y^2=4$, dobimo kandidate $(\sqrt{2},\sqrt{2})$, ~~$(-\sqrt{2},-\sqrt{2})$~~, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$.

ne ustreza $x+y \geq 0$

| $(x,y)$ | $(0,0)$ | $(\sqrt{2},\sqrt{2})$ | $(\sqrt{2},-\sqrt{2})$ | $(-\sqrt{2},\sqrt{2})$ |
|---|---|---|---|---|
| $f(x,y)$ | 0 | 6 | 2 | 2 |

najmanjša, največja vrednost f na K.