Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Schur decomposition

Let us illustrate the algorithm to find a Schur decomposition, as in § 6.1, Theorem 1.1.

Example: Find a Schur decomposition of the matrix

$$A = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}.$$

Solution: First, we want an eigenvector of A. Let us find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & -2 \\ 12 & -3-\lambda \end{vmatrix} \\ &= (7-\lambda)(-3-\lambda) + 24 \\ &= \lambda^2 - 4\lambda - 21 + 24 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

The eigenvalues are $\lambda = 1, 3$. We could arbitrarily pick one of the two and find an eigenvector, but while we're at it, let's find both:

$$\lambda = 1 : A - \lambda I = A - I = \begin{bmatrix} 6 & -2 \\ 12 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

Take $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, normalized to $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$\lambda = 3 : A - \lambda I = A - 3I = \begin{bmatrix} 4 & -2 \\ 12 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Take $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, normalized to $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

In fact, let's pick $\lambda_1 = 3$ with normalized eigenvector $u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We need to find an orthonormal basis $\{v_2\}$ of $\mathrm{Span}\{u_1\}^\perp = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}^\perp = \mathrm{Span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$. Pick $v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Now we express the transformation A in the new orthonormal basis $\{u_1, v_2\}$. Writing

$$U = \begin{bmatrix} u_1 & v_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

the matrix of A in the basis $\{u_1, v_2\}$ is

$$\begin{aligned}
[A]_{\{u_1,v_2\}} &= U^{-1}[A]_{\text{standard}}U \\
&= U^T[A]_{\text{standard}}U \\
&= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 31 & -8 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 15 & -70 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -14 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Because A was 2×2, we can take $u_2 = v_2$ and the algorithm stops here. We have found the Schur decomposition $A = UTU^*$ where

$$T = \begin{bmatrix} 3 & -14 \\ 0 & 1 \end{bmatrix}$$

is upper triangular and

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

is unitary.

Remark: The Schur decomposition is not unique, as there are choices involved in the algorithm. However, the eigenvalues of A will always appear on the diagonal of T, since A is similar to T.

The theorem does not guarantee that U and T will be real matrices, even if we start with a real matrix A.

Example: Find a Schur decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$$

Solution: First, we want an eigenvector of A. Let us find the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} \\
&= (1-\lambda)(3-\lambda) + 2 \\
&= \lambda^2 - 4\lambda + 5
\end{aligned}$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = 2 \pm \sqrt{-1} = 2 \pm i.$$

Pick $\lambda_1 = 2+i$. (The choice doesn't really matter since eigenvectors of a real matrix corresponding to complex eigenvalues come in conjugate pairs.) Let us find an eigenvector:

$$\lambda_1 = 2+i : A - \lambda_1 I = A - (2+i)I = \begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix} \sim \begin{bmatrix} 1+i & -1 \\ 0 & 0 \end{bmatrix}$$

Take $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$, normalized to $u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$. Note that for complex matrices, the null space is **not** orthogonal to the row space, but rather to the conjugate row space.

We need to find an orthonormal basis $\{v_2\}$ of

$$\begin{aligned} \mathrm{Span}\{u_1\}^\perp &= \mathrm{Span}\{\begin{bmatrix} 1 \\ 1+i \end{bmatrix}\}^\perp \\ &= \mathrm{Null} \begin{bmatrix} 1 & 1-i \end{bmatrix} \\ &= \mathrm{Span}\{\begin{bmatrix} 1-i \\ -1 \end{bmatrix}\}. \end{aligned}$$

Take the normalized vector $v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$.

Now we express the transformation A in the new orthonormal basis $\{u_1, v_2\}$. Writing

$$U = \begin{bmatrix} u_1 & v_2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

the matrix of A in the basis $\{u_1, v_2\}$ is

$$\begin{aligned} [A]_{\{u_1,v_2\}} &= U^{-1}[A]_{\text{standard}}U \\ &= U^*[A]_{\text{standard}}U \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -1+2i & 4-3i \\ 3+i & -2+i \end{bmatrix} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 6+3i & -3+6i \\ 0 & 6-3i \end{bmatrix} \\ &= \begin{bmatrix} 2+i & -1+2i \\ 0 & 2-i \end{bmatrix}. \end{aligned}$$

Because A was 2×2, we can take $u_2 = v_2$ and the algorithm stops here. We have found the Schur decomposition $A = UTU^*$ where

$$T = \begin{bmatrix} 2+i & -1+2i \\ 0 & 2-i \end{bmatrix}$$

is upper triangular and

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

is unitary.

Conclusion: Computing a Schur decomposition by hand is annoying. In practice[1], knowing the existence of a Schur decomposition is more useful than finding an explicit one.

[1] for the purposes of pure mathematics