Mathematics 1: Second midterm Cheet-Sheet (Theory)

Functions and vector functions

Function of several variables

$f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}, \vec{x} = (x_1, ..., x_n) \to f(x_1, ..., x_n)$ is a function of several variables, with domain D_f. Level curve of a function $f = f(x, y)$ is the set of all points $f(x, y) = 0$.
Partial derivative of a function w.r.t. a variable x_i at point $\vec{a}$ is $f_{x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a}) = \lim_{h \to 0} \frac{f(a_1,...,a_{i-1},a_i+h,a_{i+1},...,a_n)-f(\vec{a})}{h}$, and $(grad\ f)(\vec{a}) = [f_{x_1}(\vec{a}), ..., f_{x_n}(\vec{a})]$ is a gradient vector. The directional derivative of a function at a point in the direction of a vector $\vec{e}$ is $f_{\vec{e}}(\vec{a}) = (grad\ f)(\vec{a})\frac{\vec{e}}{||\vec{e}||} = \sum_{i=1}^n f_{x_i}(\vec{a})\frac{\vec{e}}{||\vec{e}||}$.
Second order partial derivatives:
$f_{x_i x_j}(\vec{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}) = \frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i}(\vec{x})$ from which we get the $n \times n$ Hessian matrix: $H_f(\vec{x}) = \left[\frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x})\right]_{i,j=1,...,n}$
If all second order partial derivatives $f_{x_i x_j}(\vec{x})$ are continuous at x, then the Hessian matrix is symmetric.

Vector functions of several variables

$F : D_F \subseteq \mathbb{R}^n \to \mathbb{R}^m, \vec{x} \to [f_1(\vec{x})\ ...\ f_m(\vec{x})]^T = \vec{F}(\vec{x})$.
The $m \times n$ Jacobi matrix all first order partial derivatives of $f_1, ..., f_m$ (derivative of $\vec{F}$ *over* $\vec{x}$) describes the local dilatation of the volume:

$$J_F(\vec{x}) = \frac{\partial \vec{F}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & ... & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) & ... & \frac{\partial f_2}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & ... & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix}$$

Properties: $\frac{\partial \vec{x}}{\partial \vec{x}} = I_n$, $\frac{\partial A\vec{x}}{\partial \vec{x}} = A$ if $A \in \mathbb{R}^{m \times n}$, $\frac{\partial \vec{a}^T\vec{x}}{\partial \vec{x}} = \vec{a}^T$ if $\vec{a} \in \mathbb{R}^n$, $\frac{\partial(\vec{x}^T A\vec{x})}{\partial \vec{x}} = \vec{x}^T(A + A^T)$ if $A \in \mathbb{R}^{n \times n}$, $\frac{\partial(\vec{x}^T A\vec{x})}{\partial \vec{x}} = 2\vec{x}^T A$ if $A \in \mathbb{R}^{n \times n}$ and symmetric, $\frac{\partial||\vec{x}||^2}{\partial \vec{x}} = 2\vec{x}^T$, $\frac{\partial(\vec{y}^T\vec{z})}{\partial \vec{x}} = \vec{z}^T\frac{\partial \vec{y}}{\partial \vec{x}} + \vec{y}^T\frac{\partial \vec{z}}{\partial \vec{x}}$, $\frac{\partial \vec{H}}{\partial \vec{x}} = \frac{\partial(\vec{F} \circ \vec{G})}{\partial \vec{x}} = \frac{\partial \vec{F}}{\partial \vec{G}}(\vec{G}(\vec{x}))\frac{\partial \vec{G}}{\partial \vec{x}}$

Multiple integrals

Double integral over a rectangle

$\iint_R f(x, y)\,dx\,dy = \lim_{m,n\to\infty} \sum_{j=1}^m \sum_{i=1}^n f(x^*_{ij}, y^*_{ij})\,\Delta x\,\Delta y$
where $R = [a, b] \times [c, d]$, $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{m}$ and x^*_{ij}, y^*_{ij} are chosen points in the mn smaller rectangles. This equals the volume of the bounded solid of the rectangle under the graph of $f(x, y)$ if f is nonnegative. Fubini theorem:
$\iint_R f(x, y)\,dx\,dy = \int_c^d(\int_a^b f(x, y)dx)dy = \int_a^b(\int_c^d f(x, y)dy)dx$.

Double integral

$D \subseteq \mathbb{R}^n$ is a bounded region, $f : D \to \mathbb{R}$ continuous. We choose a rectangle R such that $D \subseteq R$, and define the double integral f over region D as $\iint_D f(x, y)\,dx\,dy = \iint_R F(x, y)\,dx\,dy$, where $F(x, y) = f(x, y)$ when $(x, y) \in D$ and 0 otherwise.
Using Fubini:
If $D = \{(x, y); a \le x \le b, \varphi_1(x) \le y \le \varphi_2(x)\} \subseteq \mathbb{R}^2$ then $\iint_D f(x, y)\,dx\,dy = \int_a^b(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy)dx$.
If $D = \{(x, y); \vartheta_1(y) \le x \le \vartheta_2(y), c \le y \le d\} \subseteq \mathbb{R}^2$ then $\iint_D f(x, y)\,dx\,dy = \int_c^d(\int_{\vartheta_1(y)}^{\vartheta_2(y)} f(x, y)dx)dy$.

Triple integrals

... are defined similarly. By Fubini's theorem, we can express a triple integral as integrating three times over appropriate intervals.

Change of variables

$f : D \to \mathbb{R}$ continuous on $D \subseteq \mathbb{R}^2$ and $x = \varphi(u, v), y = \vartheta(u, v), z = \psi(u, v)$ such that $det J_{\varphi,\vartheta} \neq 0$, then: $\iint_D f(x, y)\,dx\,dy = \int_D f(\varphi(u, v), \vartheta(u, v))|det J_{\varphi,\vartheta}|\,du\,dv$ which can be extended to $\mathbb{R}^3$ getting $\iiint_D f(x, y, z)\,dx\,dy\,dz = \int_D f(\varphi(u, v, w), \vartheta(u, v, w), \psi(u, v, w))|det J_{\varphi,\vartheta,\psi}|\,du\,dv\,dw$.
Common substitutions:
Polar coordinates in $\mathbb{R}^2$ - $x = r\,cos\varphi, y = r\,sin\varphi, r \ge 0$, $\varphi \in [0, 2\pi]$, $det\,J_{polar} = r$.
Cylindrical (3D polar) coordinates in $\mathbb{R}^3 - x = r\,cos\varphi$, $y = r\,sin\varphi, z = z, r \ge 0, \varphi \in [0, 2\pi], z \in \mathbb{R}$ and $det\,J_{cylindric} = r$.
Spherical coordinates in $\mathbb{R}^3$ - $x = r\,cos\varphi\,cos\vartheta, y = r\,sin\varphi\,cos\vartheta$, $z = r\,sin\vartheta, r \ge 0, \varphi \in [0, 2\pi], \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $det\,J_{spherical} = r^2\,cos\vartheta$.

Optimisation

Classification of local extrema

$f : \mathbb{R}^n \to \mathbb{R}, \vec{a} \in \mathcal{D}_f$:
$\vec{a}$ is a local maximum of a if for all $\vec{x} \neq \vec{a}$ $(\forall||\vec{x} - \vec{a}|| < \varepsilon$ for a small $\varepsilon)$ we have $f(\vec{x}) < f(\vec{a})$.
$\vec{a}$ is a local minimum of a if for all $\vec{x} \neq \vec{a}$ $(\forall||\vec{x} - \vec{a}|| < \varepsilon$ for a small $\varepsilon)$ we have $f(\vec{x}) > f(\vec{a})$.
If f has continuous partial derivatives, then every local extremum is a critical point of f, $(grad\,f)(\vec{a}) = \vec{0}$.
If $H_f(\vec{a})$ is PD, then f has a local minimum at $\vec{a}$.
If $H_f(\vec{a})$ is ND, then f has local maximum at $\vec{a}$.
If $H_f(\vec{a})$ is indefinite, then f has no local extremum at $\vec{a}$, $\vec{a}$ is a saddle point.
If $H_f(\vec{a})$ has an eigenvalue 0 $(det\,H_f = 0)$, then we cannot conclude about the type of critical point just from $H_f(\vec{a})$.
Function is convex on $\mathcal{D}$ for all $\vec{x}, \vec{y} \in \mathcal{D}$ and all $t \in [0, 1]$ if $f(t\vec{x} + (1 - t)\vec{y}) \le t\,f(\vec{x}) + (1 - t)f(\vec{y})$. It is concave in $\mathcal{D}$ if $-f$ is convex.
For a twice differentiable $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex only if $\frac{\partial^2 f}{\partial \vec{x}^2}$ is a PSD matrix on $\mathcal{D}$, and is concave only if $\frac{\partial^2 f}{\partial \vec{x}^2}$ is NSD on $\mathcal{D}$.

Extreme values of a function subject to equality constraints

Objective: Having $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ we want to $minimise_{\vec{x}}$ $f(\vec{x})$ having constraints $g_i(\vec{x}) = 0, i = 1, 2, ..., m$ and $h_j(\vec{x}) \le 0, j = 1, 2, ...r$. We set $\mathcal{D}_{g_i} = \{x \in \mathbb{R}^n; g_i(\vec{x}) = 0\}$, $\mathcal{D}_{h_j} = \{x \in \mathbb{R}^n; g_j(\vec{x}) = 0\}$, $\mathcal{D} = \mathcal{D}_f \cap (\cap_{i=1}^m \mathcal{D}_{g_i}) \cap (\cap_{j=1}^r \mathcal{D}_{h_j})$.
The problem $(P\star)$ is now $\min_{\vec{x}\in\mathcal{D}} f(\vec{x})$.
Having only constraints $g_i(\vec{x}) = 0$, the extreme values of f are the critical points of the Lagrange function $L(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \vec{\lambda}^T\vec{G}(\vec{x}) = f(\vec{x}) - \sum_{i=1}^m \lambda_i g_i(\vec{x})$ where $\vec{G}(\vec{x}) = [g_1(\vec{x})\ ...\ g_m(\vec{x})]^T$ and $\vec{\lambda} = [\lambda_1\ ...\ \lambda_m]^T$, λ_i are Lagrange multipliers.

Dual function: Karush-Kuhn-Tucker conditions

Lagrangian: $L(\vec{x}, \vec{\lambda}, \vec{\mu}) = f(\vec{x}) - \vec{\lambda}^T\vec{G}(\vec{x}) - \vec{\mu}^T\vec{H}(\vec{x}) = f(\vec{x}) - \sum_{i=1}^m \lambda_i g_i(\vec{x}) - \sum_{j=1}^r \mu_j h_j(\vec{x})$ where $\vec{G}(\vec{x})$ and $\vec{\lambda}$ are defined as before and we define $\vec{H}(\vec{x}) = [h_1(\vec{x})\ ...\ h_r(\vec{x})]^T$, $\vec{\mu} = [\mu_1\ ...\ \mu_r]^T$. The dual function with dual variables $\vec{\lambda}$ and $\vec{\mu}$ is
$K(\vec{\lambda}, \vec{\mu}) = \inf_{\vec{x}\in\mathcal{D}} L(\vec{x}, \vec{\lambda}, \vec{\mu}) = \inf_{\vec{x}\in\mathcal{D}} \{f(\vec{x}) - \vec{\lambda}^T\vec{G}(\vec{x}) - \vec{\mu}^T\vec{H}(\vec{x})\}$.
$K(\vec{\lambda}, \vec{\mu})$ is always a concave function.
If $\mu_j \le 0$ for $j = 1, 2, ..., r$ then $K(\vec{\lambda}, \vec{\mu}) = \inf_{\vec{x}\in\mathcal{D}} L(\vec{x}, \vec{\lambda}, \vec{\mu}) \le L(\vec{x}^*, \vec{\lambda}, \vec{\mu}) = f(\vec{x}^*) - \vec{\lambda}^T\vec{G}(\vec{x}) - \vec{\mu}^T\vec{H}(\vec{x}) \le f(\vec{x}^*)$ for all $\vec{\lambda}$ and all $\vec{\mu} \le \vec{0}$.
We now have the problem $(D\star)$ to $maximize_{\vec{\lambda}\vec{\mu}}\ K(\vec{\lambda}, \vec{\mu})$ such that $\mu_j \le 0$ for $j = 1, ..., r$.
We denote $\vec{x}^*$ as the solution to $(P\star)$ and $\vec{\lambda}^*, \vec{\mu}^*$ as the solutions to $(D\star)$. Then let $p^* = f(\vec{x}^*)$ and $d^* = K(\vec{\lambda}^*, \vec{\mu}^*)$ we can note $d^* \le p^*$.
$(P\star)$ is a linear programming problem $(f(\vec{x}) = \vec{c}^T\vec{x}$ is linear, $\vec{H}(\vec{x}) = A\vec{x} - \vec{b}$ and, in addition, $\vec{x} \le 0)$.
f, h_j are convex and $\vec{G}(\vec{x}) = A\vec{x} - \vec{b}$ for some $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$ then $d^* = p^*$. In this case the optimal variables $\vec{x}^*, \vec{\lambda}^*, \vec{\mu}^*$ must satisfy KKT conditions (the first one denotes critical points of $\vec{x}^*$):

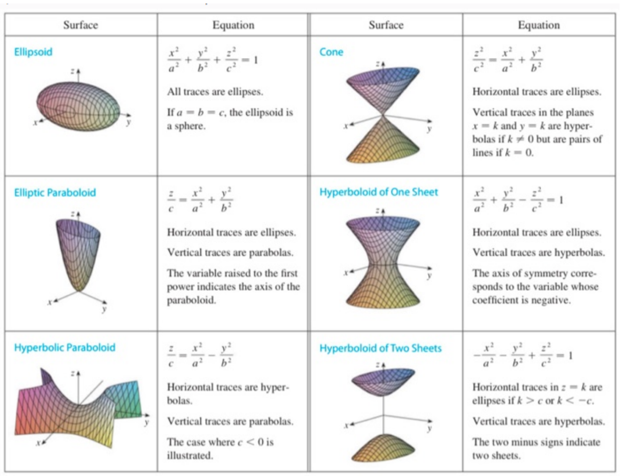
$$\inf_{\vec{x}} L(\vec{x}, \vec{\lambda}^*, \vec{\mu}^*) \to \frac{\partial L(\vec{x}, \vec{\lambda}^*, \vec{\mu}^*)}{\partial \vec{x}}(\vec{x}^*) = 0,$$
$$g_i(\vec{x}^*) = 0; i = 1, 2, ..., m,$$
$$h_j(\vec{x}^*) \le 0; j = 1, 2, ..., r,$$
$$\mu_j^* \le 0; j = 1, 2, ..., r,$$
$$\mu_j^* h_j(\vec{x}^*) = 0; j = 1, 2, ..., r$$

Other

Common trigonometry formulas

$\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$
$\sin^2(\frac{\theta}{2}) = \frac{1-\cos\theta}{2}$, $\cos^2(\frac{\theta}{2}) = \frac{1+\cos\theta}{2}$
$\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\sin(\theta \pm \gamma) = \sin\theta\cos\gamma \pm \cos\theta\sin\gamma$,
$\cos(\theta \pm \gamma) = \cos\theta\cos\gamma \pm \sin\theta\sin\gamma$
$\frac{a}{\sin\theta} = \frac{b}{\sin\gamma} = \frac{c}{\sin\delta}$
$c^2 = a^2 + b^2 - 2ab\cos\delta$, $b^2 = a^2 + c^2 - 2ac\cos\gamma$,
$a^2 = b^2 + c^2 - 2bc\cos\theta$

Common surface equations



Surface	Equation	Surface	Equation
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Common derivatives

$\frac{d}{dx}(x)=1,\ \frac{d}{dx}(|x|)=sign(x),\ \frac{d}{dx}(e^x)=e^x$,
$\frac{d}{dx}(a^x)=a^x\ln(a),\ \frac{d}{dx}(\frac{1}{x})=-\frac{1}{x^2},\ \frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}},$
$\frac{d}{dx}(\ln(f(x)))=\frac{f'(x)}{f(x)}=\frac{1}{x}\ if\ f(x)=x,\ \frac{d}{dx}(\ln|x|)=\frac{1}{x},\ x\neq 0,$
$\frac{d}{dx}(\log_a(x))=\frac{1}{xln(a)},\ x>0,\ \frac{d}{dx}(\sin(x))=\cos(x),$
$\frac{d}{dx}(\cos(x))=-\sin(x),\ \frac{d}{dx}(\tan(x))=\sec^2(x)=\tan^2(x)+1,$
$\frac{d}{dx}(\cot(x))=-\csc^2(x),\ \frac{d}{dx}(\sec(x))=\sec(x)\tan(x),$
$\frac{d}{dx}(\csc(x))=-\csc(x)\cot(x),\ \frac{d}{dx}(\sin^{-1}(x))=\frac{1}{\sqrt{1-x^2}},$
$\frac{d}{dx}(\cos^{-1}(x))=-\frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}(\tan^{-1}(x))=\frac{1}{1+x^2},$
$\frac{d}{dx}(\sinh(x))=\cosh(x),\ \frac{d}{dx}(\cosh(x))=\sinh(x),$
$\frac{d}{dx}(\tanh(x))=\frac{1}{\cosh(x)}=1-\tanh^2(x),$
$\frac{d}{dx}(\sinh^{-1}(x))=\frac{1}{\sqrt{x^2+1}},\ \frac{d}{dx}(\cosh^{-1}(x))=\frac{1}{\sqrt{x^2-1}},$
$\frac{d}{dx}(\tanh^{-1}(x))=\frac{1}{\sqrt{1-x^2}},$

Common derivatives of vector functions

For vectors x, a from $\mathbb{R}^n$, and matrix A:
$\frac{\partial x}{\partial x}=I_n$
$\frac{\partial x^Ta}{\partial x}=\frac{\partial a^Tx}{\partial x}=a$
$\frac{\partial Ax}{\partial x}=A^T$
$\frac{\partial x^TA}{\partial x}=A$
$\frac{\partial x^Tx}{\partial x}=2x$
$\frac{\partial x^TAx}{\partial x}=Ax+A^Tx$
$\frac{\partial x^TAx}{\partial x}=2Ax$, if A is symmetric
$\frac{\partial z}{\partial x}=\frac{\partial y}{\partial x}\frac{\partial z}{\partial y}$, $(z=y(x))$

Common integrals

$\int kdx=kx,\ \int x^ndx=\frac{x^{n+1}}{n+1},\ n\neq -1,\ \int\frac{1}{x^n}=\frac{-1}{(n-1)x^{n-1}},$
$\int x^{-1}dx=\int\frac{1}{x}dx=\ln|x|,\ \int a^xdx=\frac{a^x}{ln(a)},\ \int e^xdx=e^x,$
$\int\log_a(x)dx=x\log_a(x)-x\log_a(e).$
$\int\sin(x)dx=-\cos(x),\int\cos(x)dx=\sin(x)\int,\tan(x)dx=$
$-\ln|\cos(x)|=\ln|\sec(x)|,\ \int\cot(x)dx=\ln|\sin(x)|,$
$\int\sec(x)dx=\ln|\sec(x)+\tan(x)|,$
$\int\csc(x)dx=-\ln|\csc(x)+\cot(x)|,$
$\int\sin^{-1}(x)dx=x\sin^{-1}(x)+\sqrt{1-x^2},$
$\int\cos^{-1}(x)dx=x\cos^{-1}(x)-\sqrt{1-x^2},$
$\int\tan^{-1}(x)dx=x\tan^{-1}(x)-\sqrt{12}\ln(1+x^2),$
$\int\cot^{-1}(x)dx=x\cot^{-1}(x)+\sqrt{12}\ln(1+x^2),$
$\int\frac{1}{\sin(x)}dx=\ln\left|\frac{1-\cos(x)}{\sin(x)}\right|,\ \int\frac{1}{\cos(x)}dx=\ln\left|\frac{1+\sin(x)}{\cos(x)}\right|,$
$\int\frac{1}{\sin^2(x)}dx=-\cot(x),\ \int\frac{1}{\cos^2(x)}dx=\tan(x),$
$\int\frac{1}{1+\sin(x)}dx=\frac{-\cos(x)}{1+\sin(x)},\ \int\frac{1}{1+\cos(x)}dx=\frac{\sin(x)}{1+\cos(x)},$
$\int\frac{1}{1-\sin(x)}dx=\frac{\cos(x)}{1-\sin(x)},\ \int\frac{1}{1-\cos(x)}dx=\frac{-\sin(x)}{1-\cos(x)},$
$\int e^{ax}dx=\frac{1}{a}e^{ax},\ \int xe^xdx=(x-1)e^x,$
$\int xe^{ax}dx=\left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax},\ \int\frac{1}{\sqrt{x}}=2\sqrt{x},$
$\int(x+a)^ndx=\frac{(x+a)^{n+1}}{n+1},\ n\neq -1,$
$\int x(x+a)^ndx=\frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)},$
$\int\frac{ax+b}{cx+d}dx=\frac{ax}{c}-\frac{ad-bc}{c^2}\ln|cx+d|,\ \int\frac{1}{(x+a)^2}dx=-\frac{1}{x+a},$
$\int\frac{1}{ax+b}dx=\frac{1}{a}\ln|ax+b|,\ \int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right),$
$\int\sqrt{x-a}dx=\frac{2}{3}(x-a)^{\frac{3}{2}},\ \int\sqrt{ax+b}dx=\left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b},$
$\int\sqrt{x^2+a}dx=\frac{1}{2}x\sqrt{x^2+a}+\frac{a}{2}\ln|x+\sqrt{x^2+a}|,$
$\int\sqrt{a^2-x^2}dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right),$
$\int x\sqrt{x-a}dx=\frac{2}{3}a(x-a)^{\frac{3}{2}}+\frac{2}{5}(x-a)^{\frac{5}{2}},$
$\int x\sqrt{x^2\pm a^2}dx=\frac{1}{3}(x^2\pm a^2)^{\frac{3}{2}}.$

U-Substitution

The substitution, $u=g(x)$, $du=g'(x)dx$ is:

$$\int_a^b f(g(x))g'(x)dx=\int_{g(a)}^{g(b)}f(u)du$$

Integration By Parts

$$\int_a^b f(x)g'(x)dx=[f(x)g(x)]_a^b-\int_a^b f'(x)g(x)dx$$

$u=f(x),\qquad v=g(x)\ du=f'(x)dx,\quad dv=g'(x)dx$

[$\int udv=uv-\int vdu$]. As a rule of thumb use the following order, u should be the function that comes first beween: Logarthmic $\leftrightarrow$ Inverse trig. $\rightarrow$ Algebraic (Ax^n) $\rightarrow$ Trigonimetric $\rightarrow$ Exponential (k^x).

Trig-Function Trick

For $\int\sin^n(x)\cos^m(x)dx$ evaluate the following: **Deg(n) odd:** Strip one sin out and convert the rest to cos with $\sin^2(x)=1-\cos^2(x)$, then use substitution on $u=\cos(x)$. **Deg(m) odd:** Strip one cos out and convert the rest to sin with $\cos^2(x)=1-\sin^2(x)$, then use substitution on $u=\sin(x)$. **Deg(n) and Deg(m) both odd:** Use either (i) or (ii). **Deg(n) and Deg(m) both even:** Use double angle and/or half angle trig identities to reduce the integral.

Degrees	Radians	sin	cos	tan
0	0	0	1	0
30	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90	$\pi/2$	1	0	∞
120	$2\pi/3$	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$
135	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
150	$5\pi/6$	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$
180	π	0	-1	0
210	$7\pi/6$	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$
225	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1
240	$4\pi/3$	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$
270	$3\pi/2$	-1	0	∞
300	$5\pi/3$	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$
315	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
330	$11\pi/6$	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$
360	2π	0	1	0