1. Za vektorsko funkcijo $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ s predpisom $\mathbf{F}(\mathbf{r}) = \mathbf{F}(r,\varphi) = [x,y]^\mathsf{T} = \mathbf{x}$, kjer je

$$\begin{aligned} x &= r\cos\varphi,\\ y &= r\sin\varphi, \end{aligned}$$

poišči Jacobijevo matriko $J_\mathbf{F} = \frac{\partial \mathbf{F}}{\partial \mathbf{r}}$ ter pripdajočo Jacobijevo determinanto $\det(J_\mathbf{F})$.

Rešitev: $J_\mathbf{F} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$, $\det(J_\mathbf{F}) = r$.

2. Za vektorsko funkcijo $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ s predpisom $\mathbf{F}(\mathbf{r}) = \mathbf{F}(r,\varphi,z) = [x,y,z]^\mathsf{T} = \mathbf{x}$, kjer je

$$\begin{aligned} x &= r\cos\varphi,\\ y &= r\sin\varphi,\\ z &= z, \end{aligned}$$

poišči Jacobijevo matriko $J_\mathbf{F} = \frac{\partial \mathbf{F}}{\partial \mathbf{r}}$ ter pripdajočo Jacobijevo determinanto $\det(J_\mathbf{F})$.

Rešitev: $J_\mathbf{F} = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(J_\mathbf{F}) = r$.

3. Naj bo $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{r} \mapsto \mathbf{x}$ vektorska funkcija dana s predpisom $\mathbf{F}(r,\varphi,\vartheta) := [x,y,z]^\mathsf{T}$, kjer je:

$$\begin{aligned} x &= r\cos\theta\cos\varphi,\\ y &= r\cos\theta\sin\varphi,\\ z &= r\sin\theta. \end{aligned}$$

Poišči Jacobijevo matriko $J_\mathbf{F} = \frac{\partial \mathbf{F}}{\partial \mathbf{r}}$ te funkcije in pripadajočo Jacobijevo determinanto $\det(J_\mathbf{F})$.

Rešitev: $J_\mathbf{F} = \begin{bmatrix} \cos\theta\cos\varphi & -r\cos\theta\sin\varphi & -r\sin\theta\cos\varphi \\ \cos\theta\sin\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta & 0 & r\cos\theta \end{bmatrix}$, $\det(J_\mathbf{F}) = r^2\cos\theta$.

4. Naj bo $R \geq 0$, $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ pa vektorska funkcija treh spremenljivk s predpisom

$$\mathbf{F}(r,\varphi,\theta) = \mathbf{F}\left([r,\varphi,\theta]^\mathsf{T}\right) = \begin{bmatrix} (R + r\cos\theta)\cos\varphi \\ (R + r\cos\theta)\sin\varphi \\ r\sin\theta \end{bmatrix}.$$

   (a) Poišči Jacobijevo matriko funkcije $\mathbf{F}$; $J_\mathbf{F} = \frac{\partial \mathbf{F}}{\partial [r,\varphi,\theta]^\mathsf{T}}$.

   (b) Poišči determinanto zgornje Jacobijeve matrike; $\det(J_\mathbf{F})$.

Rešitev: (a) $J_\mathbf{F} = \begin{bmatrix} \cos\theta\cos\varphi & -(R + r\cos\theta)\sin\varphi & -r\sin\theta\cos\varphi \\ \cos\theta\sin\varphi & (R + r\cos\theta)\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta & 0 & r\cos\theta \end{bmatrix}$. (b) $\det(J_\mathbf{F}) = r(R + r\cos\theta)$.

① $F : \mathbb{R}^n \to \mathbb{R}^n$

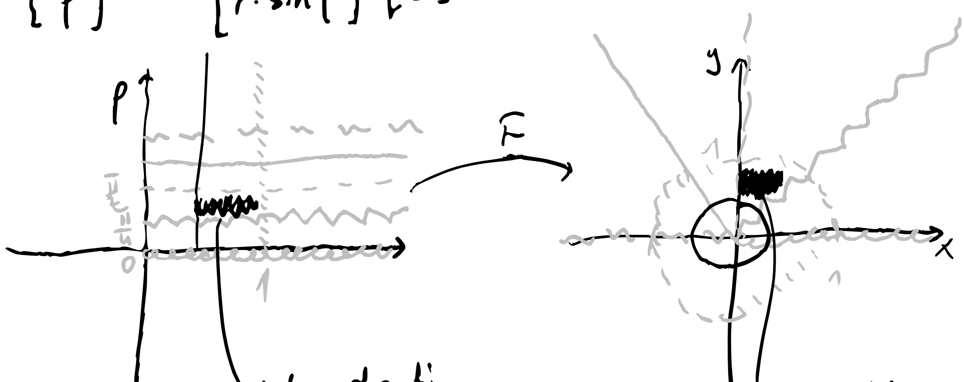
$JF : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ vrne matriko vseh parcialnih odvodov

$$F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

$$JF = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \\ & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

a) $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} r \\ \rho \end{bmatrix} \longmapsto \begin{bmatrix} r \cdot \cos \rho \\ r \cdot \sin \rho \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$ (polarne koordinate)

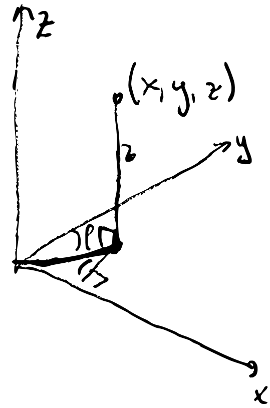


vsi kvadrati imajo enako ploščine

vsi izseki nimajo istih ploščin

$$JF = \begin{bmatrix} \cos \rho & -r \cdot \sin \rho \\ \sin \rho & r \cos \rho \end{bmatrix}$$ $\det(JF) = r \cdot \cos^2 \rho + r \sin^2 \rho = r$

② Cilindrične koordinate (v $\mathbb{R}^3$)

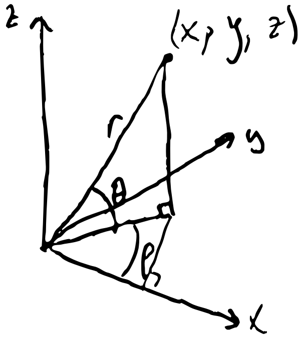


$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

$$JF = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\det(JF) = r}}$$

③ Sferične koordinate (v $\mathbb{R}^3$)



$$x = r\cos\theta\cos\varphi$$
$$y = r\cos\theta\sin\varphi$$
$$z = r\sin\theta$$

$$JF = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \text{po } r & \theta & \varphi \end{array}$$
$$JF = \begin{bmatrix} \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ \cos\theta\sin\varphi & r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$$\det(JF) = \left| \ldots \right| = r^2\cos\theta \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\cos\varphi & -\sin\varphi \\ \cos\theta\sin\varphi & -\sin\theta\sin\varphi & \cos\varphi \\ \sin\theta & \cos\theta & 0 \end{vmatrix} =$$

$$= r^2\cos\theta(-\sin\varphi|\ldots| - \cos\varphi|\ldots|) = r^2\cos\theta(-\sin^2\varphi\cdot 1 - \cos^2\varphi \cdot 1)$$

$$= -r^2\cos\theta$$

$$|\det(JF)| = r^2\cos\theta$$