1. Preslikava $\tau\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ je podana s predpisom

$$\tau(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X + X \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (a) Pokaži, da je τ linearna preslikava.

   (b) Določi njeno matriko v bazi $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ prostora $\mathbb{R}^{2\times 2}$.

Rešitev: (a) Označimo $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Potem je

$$\begin{aligned} \tau(\alpha X + \beta Y) &= A(\alpha X + \beta Y) + (\alpha X + \beta Y)A = \alpha AX + \beta AY + \alpha XA + \beta YA \\ &= \alpha(AX + XA) + \beta(AY + YA) = \alpha\tau(X) + \beta\tau(Y), \end{aligned}$$

torej je τ linearna. (b) $A_\tau = \left[\begin{smallmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{smallmatrix}\right]$.

2. Za polinom $p(x) = ax^3 + bx^2 + cx + d$ in kvadratno matriko A označimo $p(A) = aA^3 + bA^2 + cA + dI$. Naj bo $A \in \mathbb{R}^{2\times 2}$ matrika

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

   (a) Prepričaj se, da je preslikava

$$\phi\colon \mathbb{R}_3[x] \to \mathbb{R}^{2\times 2},\ \phi(p) = p(A)$$

   linearna in poišči matriko, ki ji pripada v standardnih bazah prostorov $\mathbb{R}_3[x]$ in $\mathbb{R}^{2\times 2}$.

   (b) Poišči bazo za $\ker\phi$ in določi $\dim(\ker\phi)$. (*Namig:* Če je $\Delta_A(\lambda)$ karakteristični polinom A, potem je $\Delta_A(A) = 0$.)

   (c) Naj bo $q(x) = x(x^2 - 2x - 3)$. Ali je množica vseh 2×2 matrik X, za katere velja $q(X) = 0$, vektorski podprostor v $\mathbb{R}^{2\times 2}$?

   (a) Vzemimo polinoma $p, q \in U$ in skalarja $\alpha, \beta \in \mathbb{R}$ in preverimo, da ϕ ohranja linearne kombinacije:

$$\phi(\alpha p + \beta q) = (\alpha p + \beta q)(A) = \alpha p(A) + \beta q(A) = \alpha\phi(p) + \beta\phi(q),$$

   ϕ je torej linearna. Za matriko A_ϕ poračunamo $\phi(p)$ za polinome p iz standardne baze $\mathbb{R}_3[x]$, $p \in \{1, x, x^2, x^3\}$:

$$\phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \phi(x) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \phi(x^2) = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix},\ \phi(x^3) = \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix}.$$

   Martika, ki pripada ϕ je torej

$$[\phi] = \begin{bmatrix} 1 & 1 & 5 & 13 \\ 0 & 2 & 4 & 14 \\ 0 & 2 & 4 & 14 \\ 1 & 1 & 5 & 13 \end{bmatrix}.$$

(b) Za karakteristični polinom $\Delta_A(x)$ matrike A velja $\Delta_A(A) = 0$ (ničelna matrika). Hitro vidimo

$$\Delta_A(x) = \det(A - xI) = x^2 - 2x - 3.$$

Ker noben polinom stopnje 1 ali manj ne uniči matrike A (Zakaj?), so v $\ker\phi$ le tisti polinomi stopnje ≤ 3, ki so deljivi z $\Delta_A(x)$. Od tod dobimo $\mathcal{B}_{\ker\phi} = \{x^2 - 2x - 3, x(x^2 - 2x - 3)\}$ in $\dim(\ker\phi) = 2$.

(c) Ta podmnožica *ni* vektorski podprostor. Ničle polinoma q so $x_1 = -1$, $x_2 = 0$ in $x_3 = 3$. To pomeni, da q uniči vse 2×2 matrike, ki imajo dve od teh ničel za lastni vrednosti. (Uniči sicer tudi 'polovico' od 2×2 matrik, ki imajo eno od teh ničel za dvojno lastno vrednost. Katerih ne?) Sedaj ni težko poiskati dveh matrik A in B, da velja $q(A) = 0$ in $q(B) = 0$, vendar $q(A + B) \neq 0$. (Ali matrike A, da je $q(A) = 0$, vendar $q(-A) \neq 0$.)

3. V $\mathbb{R}^3$ so dani vektorji $\mathbf{a} = [1,1,0]^\mathsf{T}$, $\mathbf{b} = [1,0,1]^\mathsf{T}$ in $\mathbf{c} = [0,1,1]^\mathsf{T}$ ter linearna preslikava $\tau : \mathbb{R}^3 \to \mathbb{R}^3$, za katero velja

$$\tau(\mathbf{a}) = \mathbf{a}, \tau(\mathbf{b}) = \mathbf{a} + \mathbf{b} \text{ ter } \tau(\mathbf{c}) = \mathbf{a} + \mathbf{c}.$$

(a) Pokaži, da je $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ baza prostora $\mathbb{R}^3$.

(b) Zapiši matriko preslikave τ v bazi $\mathcal{B} := \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

(c) Zapiši matriko preslikave τ v standardni bazi $\mathcal{S} := \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

(d) Kam preslika τ vektor $[1,1,1]^\mathsf{T}$?

Rešitev: (a) Vektorji $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{c}$ so linearno neodvisni. Ker jih je dovolj za $\dim \mathbb{R}^3 = 3$, tvorijo bazo. (b) $A_{\tau,\mathcal{B},\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $A_{\tau,\mathcal{S},\mathcal{S}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, (d) $\tau([1,1,1]^\mathsf{T}) = [2,2,1]^\mathsf{T}$.

4. Naj bo $\mathbb{R}_3[x]$ vektorski prostor polinomov p stopnje kvečjemu 3.

(a) Prepričaj se, da je preslikava $\phi\colon \mathbb{R}_3[x] \to \mathbb{R}^3$, $\phi(p) := [p(-1), p(0), p(1)]^\mathsf{T}$ linearna.

(b) Poišči bazo $\mathcal{B}_{\ker\phi}$ jedra $\ker\phi$ preslikave ϕ.

(c) Zapiši matriko, ki pripada ϕ v bazi $\{1, x, x^2, x^3\}$ za $\mathbb{R}_3[x]$ in standardni bazi $\mathbb{R}^3$.

Rešitev: (b) $\mathcal{B}_{\ker\phi} = \{x^3 - x\}$, $\mathcal{B}_{\operatorname{im}\phi} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, (c) $A_\phi = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

5. Dana je preslikava $\psi : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$

$$(\psi(p))(x) = (xp(x+1))' - 2p(x).$$

Pokaži, da je ψ linearna. Poišči njeno matriko v bazi $\{1, x, x^2\}$ ter njeno jedro in sliko.
Rešitev: $A_\psi = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $\ker\psi = \{a(x+1) : a \in \mathbb{R}\}$, $\mathcal{B}_{\ker\psi} = \{x+1\}$,
$\operatorname{im}\psi = \{a + 4bx + bx^2 : a, b \in \mathbb{R}\}$, $\mathcal{B}_{\operatorname{im}\psi} = \{1, x^2 + 4x\}$.

6. Naj bo $\mathbf{a} = [1,1]^\mathsf{T}$. Preslikava $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ je dana s predpisom

$$\phi(\mathbf{x}) = \mathbf{x}\,\mathbf{a}^\mathsf{T} = \mathbf{x}\,[1,1].$$

   (a) Utemelji, da je ϕ linearna preslikava.
   (b) Poišči matriko, ki pripada ϕ v standardnih bazah prostorov $\mathbb{R}^2$ in $\mathbb{R}^{2\times 2}$.
   (c) Določi $\dim(\ker\phi)$ in $\dim(\operatorname{im}\phi)$.
   (d) Poišči bazo za $\operatorname{im}\phi$.

Rešitev: (a) Preverimo, da ϕ ohranja linearne kombinacije. Velja

$$\phi(\alpha\mathbf{x}+\beta\mathbf{y}) = (\alpha\mathbf{x}+\beta\mathbf{y})\mathbf{a}^\mathsf{T} = \alpha\mathbf{x}\,\mathbf{a}^\mathsf{T} + \beta\mathbf{y}\,\mathbf{a}^\mathsf{T} = \alpha\phi(\mathbf{x}) + \alpha\phi(\mathbf{y})$$

za vse $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$ in vse $\alpha,\beta \in \mathbb{R}$, ϕ je linearna.

(b) Poračunajmo, v kaj ϕ preslika vektorje standardne baze $\mathbb{R}^2$.

$$\phi\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}[1,1] = \begin{bmatrix}1&1\\0&0\end{bmatrix} = E_{11}+E_{12}$$
$$\phi\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}[1,1] = \begin{bmatrix}0&0\\1&1\end{bmatrix} = E_{21}+E_{22}$$

Matrika, ki pripada ϕ je torej

$$A_\phi = \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}.$$

(c) Matrika A_ϕ ima rang 2, torej $\dim(\operatorname{im}\phi) = \dim(C(A_\phi)) = 2$. Za jedro potem velja $\dim(\ker\phi) = \dim(\mathbb{R}^2) - \dim(\operatorname{im}\phi) = 2-2 = 0$.

(d) Zagotovo sta v $\operatorname{im}\phi$ matriki $\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$. Ker sta linearno neodvisni in je $\dim(\operatorname{im}\phi) = 2$, je baza za $\operatorname{im}\phi$ kar $\mathcal{B}_{\operatorname{im}\phi} = \left\{\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]\right\}$.

7. Naj bosta U in V vektorska podprostora v vektorskem prostoru W. Definiramo množice:

$$U\times V := \{(u,v) : u\in U, v\in V\},$$
$$U+V := \{u+v : u\in U, v\in V\} \text{ ter}$$
$$U\cap V := \{w\in W : w\in U \text{ in } w\in V\}.$$

   (a) Prepričaj se, da sta $U+V$ in $U\cap V$ vektorska podprostora v W.
   (b) 'Ugani' ustrezno strukturo vektorskega prostora na $U\times V$. Utemelji, da je v tem primeru $U\times V$ res vektorski prostor! Kako lahko $\dim(U\times V)$ izraziš z $\dim U$ in $\dim V$?
   (c) Preslikava $\phi\colon U\times V \to W$ naj bo dana s $\phi(u,v) = u-v$. Prepričaj se, da je ϕ linearna. (Če ni, se vrni k točki (b) te naloge.) Določi $\ker\phi$ in $\operatorname{im}\phi$.

(d) Preveri, da je preslikava $\psi\colon U \cap V \to \ker\phi$, $\psi(w) = (w,w)$ linearna in bijektivna, torej velja $\dim(U \cap V) = \dim(\ker\phi)$.

(e) Zaključi, da velja $\dim U + \dim V = \dim(U + V) + \dim(U \cap V)$.

Rešitev: (b) Seštevanje in množenje s skalarjem definiramo na 'očiten' način:

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2),$$
$$\alpha(u,v) := (\alpha u, \alpha v),$$

Za dimenzijo velja $\dim(U \times V) = \dim U + \dim V$.
(c) $\ker\phi = \{(w,w) : w \in U \cap V\}$, $\operatorname{im}\phi = U + V$.
(e) Iz dimenzijske enačbe

$$\dim(\ker\phi) + \dim(\operatorname{im}\phi) = \dim(U \times V)$$

sledi

$$\dim(U \cap V) + \dim(U + V) = \dim U + \dim V.$$

$(\mathbb{R}_n[x], +, \cdot)$ realni polinomi v x do stopnje n

$(p+q)(x) := p(x) + q(x)$

$(\alpha p)(x) := \alpha p(x)$

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \iff (a_n, a_{n-1}, \dots, a_1, a_0)$

$q(x) = b_n x^n + \dots + b_1 x + b_0 \iff (b_n, b_{n-1}, \dots, b_1, b_0)$

$(p+q)(x) = (a_n+b_n)x^n + \dots + (a_1+b_1)x + a_0 + b_0 \iff (a_n+b_n, \dots, a_0+b_0)$

Standardna baza: monomi

$x^n \iff (1, 0, \dots, 0) = e_n$

$1 \iff (0, \dots, 0, 1)$

? a) $U_1 = \{ p(x) = ax + b;\ \text{kjer je } a \neq 0 \}$

prejšnji teden

$U_1 \subseteq \mathbb{R}_1[x]$

$0 \notin U_1 \Rightarrow$ ni VPP

b) $U_2 = \{ p;\ p(1) = 0 \}$ Vsi polinomi, ki grejo skozi $(1, 0)$.

$U_2 \in \mathbb{R}_2[x]$

Vsota? ✓ Mn. s skalarjem? ✓

$p, q \in U_2 \iff p(1) = q(1) = 0$

$(\alpha p + \beta q)(1) = (\alpha p)(1) + (\beta q)(1) = 0$ $\boxed{✓}$

Pa baza?

1. način $p(x) = ax^2 + bx + c$ poljuben kvadratni polinom

$p(1) = 0$

$\Downarrow$

$a + b + c = 0$

$[1 \ 1 \ 1 \ \vdots \ 0]$

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Updownarrow$ $\Updownarrow$

$p_1(x) = -x^2 + x$ $p_2(x) = -x^2 + 1$

2. način $p(1) = 0$ ker je ničla v 1

$p(x) = (x-1)(Ax + B)$

$\Downarrow$

$q_1(x) = (x-1)x$

$q_2(x) = (x-1)1$

c) $\mathcal{U}_3 = \{p : p(0) = 1\}$

Ni VPP: množenje s skalarjem $\times$

in $0 \notin \mathcal{U}_3$

in ni zaprto za seštevanje

d) $\mathcal{U}_4 = \{p : p''(3) = 0\}$

$\mathcal{U}_4 \in \mathbb{R}_3[x]$

če $p, q \in \mathcal{U}_4 : p''(3) = 0$

$q''(3) = 0$

$\Rightarrow (\alpha p + \beta q)''(3) =$

$= \alpha p''(3) + \beta q''(3) = 0$

Baza?

1. $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

$p''(3) = 18a + 2b = 0$

$9a + b = 0$

$\mathcal{U}_4 = N([9 \ 1 \ 0 \ 0])$

$v_1 = \begin{bmatrix} -1 \\ 9 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$p_1(x) = -x^3 + 9x^2 \quad p_2(x) = x \quad p_3(x) = 1$

2. $p''(x) = (x-3)A$

$p'(x) = A\frac{x^2}{2} - 3Ax + B$ } integriramo

$p(x) = \frac{A}{6}x^3 - \frac{3}{2}Ax^2 + Bx + C$

$q_1(x) = x^3 - 9x^2$

$q_2(x) = x$

$q_3(x) = 1$

podobni, a?

Lin. preslikave

$f : U \to V$

f je linearna, če

① za $\forall u, v \in V$ velja $f(u+v) = f(u) + f(v)$

② Za $\forall u \in V$ in $\alpha \in \mathbb{R}$ velja $f(\alpha u) = \alpha f(u)$

Primer: $A \in \mathbb{R}^{n \times m}$

$L_A(\vec{x}) = A\vec{x}$

$L_A : \mathbb{R}^m \to \mathbb{R}^n$

$L_A(\alpha\vec{x} + \beta\vec{y}) = A(\alpha\vec{x} + \beta\vec{y}) = A(\alpha\vec{x}) + A(\beta\vec{y}) =$

$= L_A(\alpha\vec{x}) + L_A(\beta\vec{y})$

6. $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$ $\quad \phi: \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$

$\phi(x) = xa^T \qquad \begin{bmatrix}\cdot\\\cdot\end{bmatrix}\begin{bmatrix}\cdot & \cdot\end{bmatrix} = \begin{bmatrix}\cdot & \cdot\\ \cdot & \cdot\end{bmatrix}$

a) ϕ je linearna

$$\phi(\alpha x + \beta y) = (\alpha x + \beta y)a^T = \alpha x a^T + \beta y a^T = \\ = \alpha\phi(x) + \beta\phi(y)$$

b) Predstavi ϕ z matriko v standardnih bazah $\mathbb{R}^2$ in $\mathbb{R}^{2\times 2}$.

Baza $\mathbb{R}^2$: $\{e_1, e_2\}$

Baza $\mathbb{R}^{2\times 2}$: $\{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$\phi(e_1) = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = 1\cdot E_{11} + 1\cdot E_{12}$$

$$\phi(e_2) = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} = 1\cdot E_{21} + 1\cdot E_{22}$$

Matrika, ki predstavlja preslikavo ϕ:

$$[\phi]_{\mathcal{S},\mathcal{S}} = \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix} \end{array} \begin{array}{c} E_{11}\\ E_{12}\\ E_{21}\\ E_{22}\end{array} = M$$

c) Jedro? Im?

$\ker(\phi) = \{x : \phi(x) = 0\} = N(M)$

če imamo izbrane baze ??

$M \sim \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}$ Dobimo matr. polnega ranga

$\Rightarrow N(M)$ je le ničelni vektor $= \{0\}$

$\mathrm{im}(\phi) = \{y : \phi(x) = y \text{ za nek } x\} = C(M)$

$= \mathrm{Lin}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$??? $\quad \mathrm{im}\,\phi = \mathrm{Lin}\left(\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right)$

4 $\mathbb{R}_3[x]$

hibridni razsmernik

$$\phi: \mathbb{R}_3[x] \to \mathbb{R}^3$$

$$\phi(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

a) ϕ linearna

uporabimo definicijo ϕ

uporabimo def. seštevanja polinomov in vektorjev

$$\phi(\alpha p + \beta q) = \begin{bmatrix} (\alpha p + \beta q)(-1) \\ (\alpha p + \beta q)(0) \\ (\alpha p + \beta q)(1) \end{bmatrix} = \alpha \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \beta \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} =$$

$$= \alpha \phi(p) + \beta \phi(q) \checkmark$$

b) Poišči matriko ϕ iz std. baze v std. bazo.

Baza $\mathbb{R}_3[x]$:

$p_0(x) = 1$

$p_1(x) = x$

$p_2(x) = x^2$

$p_3(x) = x^3$

$$\phi(p_0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \phi(p_1) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \phi(p_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \phi(p_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$M = \begin{array}{cc} \begin{array}{cccc} \phi(p_0) & \phi(p_1) & \phi(p_2) & \phi(p_3) \end{array} & \\ \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

Poišči jedro in im!

$$M \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C(M) = \mathcal{L}(\phi(p_0), \phi(p_1), \phi(p_2)) = \mathbb{R}^3$$

$\operatorname{im} \phi = \mathbb{R}^3$ tj., kot rezultat lahko dobimo katerikoli vektor

$a = 0$

$b + d = 0$

$c = 0$

baza za $N(M) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\updownarrow$

$(0p_0 + (-1)p_1 + 0p_2 + 1p_3)(x) = -x + x^3$

Baza za $\ker \phi = \{x^3 - x\}$

Lažje: $\phi(p) = 0$

$$\begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad p(x) = (x+1)x \cdot (x-1) \cdot A$$