1. Izračunaj spodnje dvojne integrale.

   (a) $\iint_D (5-x-y)\,dxdy$, kjer je $D=[0,1]\times[0,1]$,

   (b) $\iint_D \frac{y}{x+1}\,dxdy$, kjer je D določeno z $x\geq 0$, $y\geq x$ in $x^2+y^2\leq 2$,

   (c) $\iint_D \frac{\sin x}{x}\,dxdy$, kjer je D trikotnik določen z $0\leq y\leq x$ in $x\leq\pi$,

   (d) $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy$ in s pomočjo tega izračunaj $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

   Rešitev: (a) 4, (b) $\frac{1}{2}$, (c) 2, (d) Uvedemo polarne koordinate, dobimo π in $\sqrt{\pi}$.

2. Skiciraj integracijsko območje in izračunaj dvakratna integrala.

   (a) $\int_0^1\left(\int_{-x}^{x} xe^y\,dy\right)dx$,

   (b) $\int_0^1\left(\int_0^y \frac{y}{x+1}dx\right)dy+\int_1^{\sqrt{2}}\left(\int_0^{\sqrt{2-y^2}} \frac{y}{x+1}dx\right)dy$.

   Rešitev: (a) $\frac{2}{e}$, (b) $\frac{1}{2}$.

3. Izračunaj prostornino telesa, ki je omejeno s paraboloidom $z=8-x^2-y^2$ in ravnino $z=-1$.
   Rešitev: $V=\frac{81\pi}{2}$.

4. Poišči koordinate masnega središča četrtine kroga; $x^2+y^2\leq R^2$, $x\geq 0$, $y\geq 0$, če je gostota v vsaki točki enaka oddaljenosti od izhodišča, tj. $\rho(x,y)=\sqrt{x^2+y^2}$.
   Namig: masa lika $D\subseteq\mathbb{R}^2$ je dana z $m=\iint_D \rho(x,y)dxdy$, koordinati masnega središča pa sta $x^*=\frac{1}{m}\iint_D x\rho(x,y)dxdy$ in $y^*=\frac{1}{m}\iint_D y\rho(x,y)dxdy$. Uvedi polarne koordinate.
   Rešitev: $x^*=y^*=\frac{3R}{2\pi}$.

5. Določi maso in koordinate masnega središča homogenega telesa (tj. $\rho(x,y,z)=1$), ki je omejeno s ploskvama $z^2=x^2+y^2$ ter $x^2+y^2+z^2=4$ in leži v polprostoru $z\geq 0$.
   Namig: Vpelji ti. sferne oz. krogelne koordinate:

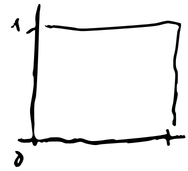
$$x=r\cos\theta\cos\varphi,$$
$$y=r\cos\theta\sin\varphi,$$
$$z=r\sin\theta,$$

   tj. 'novo spremenljivko' $\mathbf{F}(r,\varphi,\theta)=[x,y,z]^{\mathsf{T}}$ (za katero je $\det(J\mathbf{F})=r^2\cos\theta$.)
   Rešitev: $m=\frac{8\pi}{3}(2-\sqrt{2})$, $x^*=y^*=0$, $z^*=\frac{3}{8}(2+\sqrt{2})$.

6. Določi maso in koordinate masnega središča krogle z neenačbo $x^2+y^2+z^2\leq 2z$, če je njena gostota v vsaki točki enaka oddaljenosti od izhodišča.
   Namig: Uvedi krogelne koordinate.
   Rešitev: $m=\frac{8\pi}{5}$, $x^*=y^*=0$, $z^*=\frac{8}{7}$.

7. Telo $D \subseteq \mathbb{R}^3$ je omejeno s paraboličnima valjema $z = 2 - x^2$ in $z = y^2 - 2$. Izračunaj prostornino in maso tega telesa, če je gostota enaka $\rho(x, y, z) = y^2$.
Namig: Poišči (pravokotno) projekcijo tega telesa na xy-ravnino, uvedi cilindrične koordinate.
Rešitev: $V = 8\pi$, $m = \frac{16\pi}{3}$.

1 a) $\iint\limits_D (5-x-y)\,dx\,dy \qquad D=[0,1]\times[0,1]$

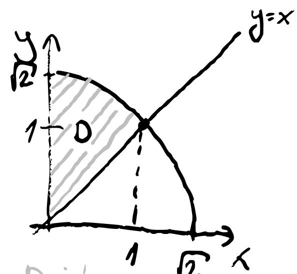


$$=\int_{y=0}^{1}\left(\int_{x=0}^{1}(5-x-y)\,dx\right)dy=$$

$$=\int_{y=0}^{1}\left(5x-\frac{x^2}{2}-yx\right)\Big|_{x=0}^{1}dy=\int_0^1\left(5-\frac{1}{2}-y\right)dy=$$

$$=\int_0^1\left(\frac{9}{2}-y\right)dy=\left(\frac{9}{2}y-\frac{y^2}{2}\right)\Big|_0^1=\frac{9}{2}-\frac{1}{2}=4$$

b) $\iint\limits_D \frac{y}{x+1}\,dx\,dy \qquad D=\{(x,y);\ x\geq 0,\ y\geq x,\ x^2+y^2\leq 2\}$



1. NAČIN zamislimo si, da je x fiksiran, gledamo ipsilone nad njim -- meje?

$$=\int_{x=0}^{1}\left(\int_{y=x}^{\sqrt{2-x^2}}\frac{y}{x+1}\,dy\right)dx$$

2. NAČIN če gledamo y -- dva dela D-ja! $y\in[0,1)$ in $y\in[1,\sqrt{2}]$. Razdelimo! D_1 in D_2

$$=\underbrace{\int_{y=0}^{1}\left(\int_{x=0}^{y}\frac{y}{x+1}\,dx\right)dy}_{1}+\underbrace{\int_{y=1}^{\sqrt{2}}\left(\int_{x=0}^{\sqrt{2-y^2}}\frac{y}{x+1}\,dx\right)dy}_{2}$$

3. NAČIN opišemo s polarnimi koord.

$r\in[0,\sqrt{2}],\ \varphi\in\left[\frac{\pi}{4},\frac{\pi}{2}\right] \qquad D=[0,\sqrt{2}]\times\left[\frac{\pi}{4},\frac{\pi}{2}\right] \qquad \begin{aligned}x&=r\cos\varphi\\ y&=r\sin\varphi\end{aligned}$

$$\ldots=\int_{r=0}^{\sqrt{2}}\left(\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{r\sin\varphi}{r\cos\varphi+1}\cdot\underbrace{r}_{|\det J|}\,d\varphi\right)dr$$

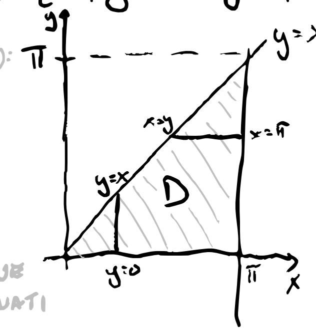
Jac.matr.

NADALJUJMO S <u>1. NAČINOM</u>.

$$=\int_{x=0}^{1}\left(\frac{1}{x+1}\cdot\frac{y^2}{2}\right)\Big|_{y=x}^{\sqrt{2-x^2}}dx=\int_{x=0}^{1}\frac{1}{2(x+1)}\left(2-x^2-x^2\right)dx=\int_{x=0}^{1}\frac{1-x^2}{1+x}\,dx=$$

$$=\int_{x=0}^{1}(1-x)\,dx=\left(x-\frac{x^2}{2}\right)\Big|_0^1=1-\frac{1}{2}=\frac{1}{2}$$

c) $\iint \frac{\sin x}{x}\,dx\,dy$ $\qquad D = \{(x,y) : 0 \le y \le x;\ x \le \pi\}$



Za dan y razmislimo: kakšne so meje x?

$$= \int_{y=0}^{\pi}\left(\int_{x=y}^{\pi} \frac{\sin x}{x}\,dx\right)dy =$$

TEGA SE NE DA IZRAČUNATI

$$= \int_{x=0}^{\pi}\left(\int_{y=0}^{x}\frac{\sin x}{x}\,dy\right)dx = \int_{x=0}^{\pi}\left(\frac{\sin x}{x}\cdot y\right)\Big|_{y=0}^{x} dx =$$

$$= \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 1-(-1) = 2$$

d) $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = ?$

Tudi nedol. int. te funkcije se ne da izraziti z elementarno funkcijo. Toda - trik!

$$\iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy = \dots$$

$$\boxed{\text{polarne koordinate: } x^2+y^2 = r^2, \quad \left.\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}\right\}}$$

$$\dots = \int_{r=0}^{\infty}\left(\int_{\varphi=0}^{2\pi} e^{-\frac{r^2}{2}} \cdot \underbrace{r}_{|\det J|} \cdot d\varphi\right)dr = 2\pi\int_{r=0}^{\infty} e^{-\frac{r^2}{2}}\, r\,dr$$

zaradi tega je lažje izračunati

$$\left[\int_{x=a}^{b}\left(\int_{y=c}^{d} f(x)g(y)\,dy\right)dx = \int_{x=a}^{b} f(x)\left(\int_{y=c}^{d} g(y)\,dy\right)dx = \left(\int_{y=c}^{d} g(y)\,dy\right)\left(\int_{x=a}^{b} f(x)\,dx\right)\right]$$

$$t = \frac{r^2}{2} \Rightarrow dt = r\,dr$$

$$= 2\pi\int_{t=0}^{\infty} e^{-t}\,dt = 2\pi\left(-e^{-t}\right)\Big|_0^{\infty} = 2\pi(-0-(-1)) = 2\pi$$

izračunamo limito

Toda: kakšna je zveza z originalnim integralom?

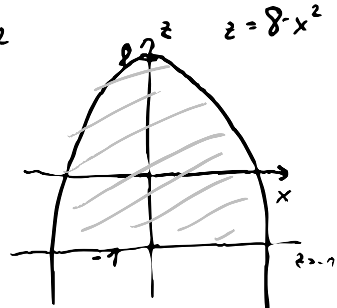
$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-\frac{x^2}{2}}\cdot e^{-\frac{y^2}{2}}\,dx\,dy =$$

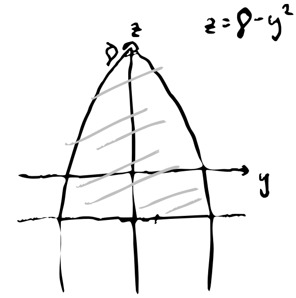
$$= \left(\int_{x=-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\right)\left(\int_{y=-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy\right) = \left(\int_{x=-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\right)^2$$

$$\dots \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

③ Telo, omejeno z: Prostornina?

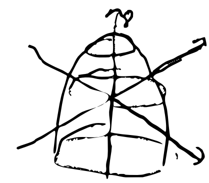
- $z = 8 - x^2 - y^2$
- $z = -1$.





Torej si lahko predstavljamo, da vzamemo to parabolo in jo zavrtimo okrog svoje osi.

Volumen: $V(D) = \iiint_D 1 \cdot dx\,dy\,dz$



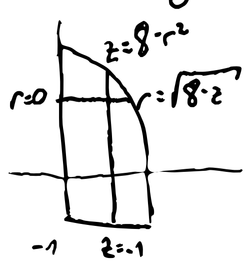
Cilindrične koordinate

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$

$|\det JF| = r$

Pri $z = -1$: računamo y
$z = 8 - y^2 \Rightarrow -1 = 8 - y^2 \Rightarrow y = \pm 3$

$z = 8 - x^2 - y^2 = 8 - r^2$



$\varphi \in [0, 2\pi]$

$r \in [0, 3]$ in $z \in [-1, 8 - r^2]$

ALI

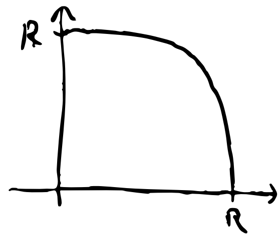
$r \in [0, \sqrt{8 - z}]$ in $z = [-1, 8$

$$\iiint_D 1\,dx\,dy\,dz = \int_{\varphi=0}^{2\pi}\left(\int_{r=0}^{3}\left(\int_{z=-1}^{8-r^2} 1 \cdot r\,dz\right)dr\right)d\varphi =$$

$$= 2\pi \int_{r=0}^{3}\left(rz\Big|_{z=-1}^{8-r^2}\right)dr = 2\pi\int_{r=0}^{3}\left(r(8-r^2)+r\right)dr =$$

$$= 2\pi\int_0^3 (9r - r^3)\,dr = 2\pi\left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Big|_0^3 = 2\pi\,\frac{81}{4} = \frac{81\pi}{2}$$

(4) $D = \{(x,y);\ x^2+y^2 \leq R^2,\ x \geq 0,\ y \geq 0\}$



Gostota $\rho(x,y) = \sqrt{x^2+y^2} = r$

(narašča proti robu)

$m = \iint_D \rho(x,y)\,dx\,dy$ masa $= \int_{\varphi=0}^{\frac{\pi}{2}} \left(\int_{r=0}^{R} r \cdot r\,dr \right) d\varphi = \frac{\pi}{2} \cdot \left(\frac{r^3}{3} \right) \Big|_0^R = \frac{\pi R^3}{6}$

Jac.

$x^* = \frac{1}{m} \iint_D x\,\rho(x,y)\,dx\,dy = \frac{1}{m} \int_{\varphi=0}^{\frac{\pi}{2}} \left(\int_{r=0}^{R} r\cos\varphi \cdot r \cdot r\,dr \right) d\varphi = \frac{1}{m} \left(\int_{\varphi=0}^{\frac{\pi}{2}} \cos\varphi\,d\varphi \right) \left(\int_0^R r^3\,dr \right)$

težišči

$y^* = \frac{1}{m} \iint_D y\,\rho(x,y)\,dx\,dy$

$= \frac{1}{m} (\sin\varphi) \Big|_0^{\frac{\pi}{2}} \left(\frac{r^4}{4} \right) \Big|_0^R = \frac{1}{m} \cdot 1 \cdot \frac{R^4}{4} = \frac{3R}{2\pi}$

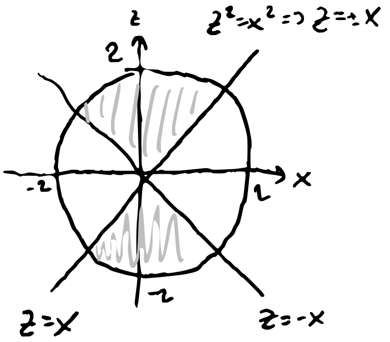
"enako kot x^* ... logično zaradi simetrije

Koordinata težišča:

$(x^*, y^*) = \left(\frac{3R}{2\pi}, \frac{3R}{2\pi} \right)$

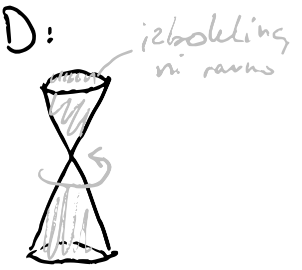
③ $\rho(x,y,z)=1$

$z^2=x^2+y^2$ in $x^2+y^2+z^2=4$
↳ sfera s polmerom 2



In enako za ravnino $\overset{z}{\uparrow}\!\!\to y$.

D:



Sferne koordinate

$x=r\cos\vartheta\cos\varphi$

$y=r\cos\vartheta\sin\varphi$

$z=r\sin\vartheta$

$|\det(JF)|=r^2\cos\vartheta$

$r\in[0,2]$

$\vartheta\in[\frac{\pi}{4},\frac{\pi}{2}]$

$\varphi\in[0,2\pi]$

Če zapišemo enačbo
$x^2+y^2=z^2$
v sfernih koordinatah, dobimo
$r^2\cos^2\vartheta=r^2\sin^2\vartheta$
$\cos\vartheta=\sin\vartheta$

$$m=\int_{r=0}^{2}\int_{\vartheta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi} r^2\cos\vartheta\, d\varphi\, d\vartheta\, dr=$$

Če je gostota 1, je masa isto kot volumen

$$=\left(\int_{\varphi=0}^{2\pi}1\,d\varphi\right)\left(\int_{\vartheta=\pi/4}^{\pi/2}\cos\vartheta\, d\vartheta\right)\left(\int_{r=0}^{2}r^2dr\right)=$$

$$=2\pi\cdot\sin\vartheta\Big|_{\pi/4}^{\pi/2}\cdot\frac{r^3}{3}\Big|_0^2=2\pi\left(1-\frac{\sqrt{2}}{2}\right)\frac{8}{3}=\frac{8\pi}{3}(2-\sqrt{2})$$

$$x^*=\frac{1}{m}\iiint_D x\,\rho(x,y,z)\,dx\,dy\,dz$$

Težišče

$$y^*=\frac{1}{m}\iiint_D y\,\rho(x,y,z)\,dx\,dy\,dz$$

$$z^*=\frac{1}{m}\iiint_D z\,\rho(x,y,z)\,dx\,dy\,dz$$

$$z^*=\frac{1}{m}\int_{r=0}^{2}\int_{\vartheta=\frac{\pi}{4}}^{\pi/2}\int_{\varphi=0}^{2\pi} r^3\sin\vartheta\cos\vartheta\, d\varphi\, d\vartheta\, dr=$$

$$=\frac{1}{m}\int_{\varphi=0}^{2\pi}1\,d\varphi\int_{r=0}^{2}r^3dr\int_{\vartheta=\pi/4}^{\pi/2}\sin\vartheta\underbrace{\cos\vartheta\, d\vartheta}=$$

$$\boxed{\begin{array}{l}t=\sin\vartheta\\ dt=\cos\vartheta\, d\vartheta\end{array}}$$

$$=\frac{1}{m}\cdot2\pi\cdot\frac{r^4}{4}\Big|_0^2\cdot\int_{t=\frac{\sqrt{2}}{2}}^{1}t\,dt=\frac{1}{m}\cdot2\pi\cdot\frac{16}{4}\cdot\frac{t^2}{2}\Big|_{\frac{\sqrt{2}}{2}}^{1}=$$

$$=\frac{4\pi}{m}\left(1-\frac{1}{2}\right)=\frac{2\pi}{m}=\frac{3}{4(2-\sqrt{2})}=\frac{3}{8}(2+\sqrt{2})$$

$$y^*=\frac{1}{m}\int_{r=0}^{2}\int_{\vartheta=\frac{\pi}{4}}^{\pi/2}\int_{\varphi=0}^{2\pi} r^3\cos^2\vartheta\sin\varphi\, d\varphi\, d\vartheta\, dr$$

$$=\frac{1}{m}\underbrace{\int_{\varphi=0}^{2\pi}\sin\varphi\, d\varphi}_{0}$$

⑥ $x^2+y^2+z^2 \le 2z$ $\quad \rho(x,y,z)=\sqrt{x^2+y^2+z^2}=r$

uporabimo sferične koordinate

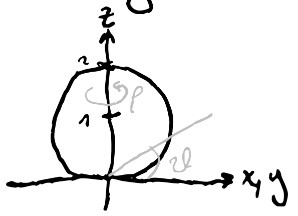
Določimo maso in težišče.

Preuredimo enačbo:

$$x^2+y^2+z^2-2z \le 0 \qquad /+1$$

$$x^2+y^2+(z-1)^2 \le 1$$

Krogla s središčem v $(0,0,1)$ in polmerom 1.

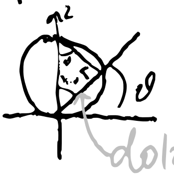


Uporabili bomo sferične koord.

$\varphi \in [0, 2\pi]$ sim. okoli z osi

$\vartheta \in [0, \pi/2]$

Povedati pa bomo morali meje za r pri danem ϑ.



dolžina: $2\cdot\sin\vartheta$

$$\vartheta\in[0,\pi/2] \Rightarrow r\in[0, 2\sin\vartheta]$$

Ali iz enačbe:

$$0 \le r^2 \le 2r\sin\vartheta$$

$$0 \le r \le 2\sin\vartheta$$

$$m = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi/2}\int_{r=0}^{2\sin\vartheta} r^3\cos\vartheta \, dr\, d\vartheta\, d\varphi =$$

$$= 2\pi \int_{\vartheta=0}^{\pi/2} \cos\vartheta \, \frac{r^4}{4}\Big|_0^{2\sin\vartheta} d\vartheta = \frac{2\pi\cdot 16}{4}\int_{\vartheta=0}^{\pi/2} \cos\vartheta \sin^4\vartheta \, d\vartheta$$

$t=\sin\vartheta$

$dt=\cos\vartheta\, d\vartheta$

$$= \frac{\pi}{2}\cdot 16\int_0^1 t^4\,dt = \frac{\pi\cdot 16}{2}\,\frac{t^5}{5}\Big|_0^1 = 8\pi\cdot\frac{1}{5}$$

Zaradi sim. sklepamo, da bosta x^* in y^* enaka 0.

$$z^* = \frac{1}{m}\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi/2}\int_{r=0}^{2\sin\vartheta} r^4 \sin\vartheta\cos\vartheta\, dr\, d\vartheta\, d\varphi =$$

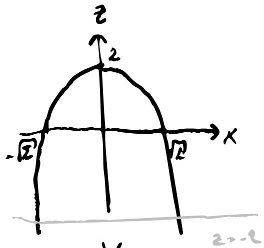
$$= \frac{1}{m}\, 2\pi \int_{\vartheta=0}^{\pi/2} \sin\vartheta\cos\vartheta \left(\frac{r^5}{5}\right)\Big|_0^{2\sin\vartheta} d\vartheta =$$

$$= \frac{1}{m}\,\frac{2\pi}{5}\cdot 32\cdot\int_{\vartheta=0}^{\pi/2} \sin^6\vartheta\cos\vartheta\, d\vartheta = \frac{1}{m}\,\frac{64\pi}{5}\int_0^1 t^6\,dt =$$

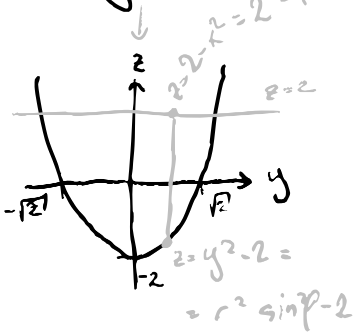
$$= \frac{64\pi}{m 5}\cdot\frac{1}{7} = \frac{8}{7}$$

7 Ploskvi $\rho(x,y,z)=y^2$

$z=2-x^2$ in $z=y^2-2$

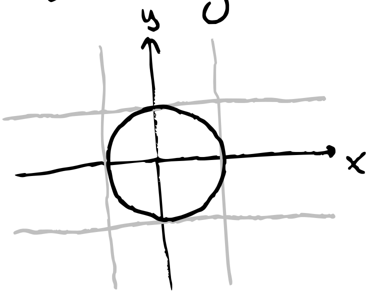


za $\forall y$



"Pravokotna žlebova"

Presek v xy ravnini:



$2-x^2=y^2-2$

$x^4+y^2=4$ krožnica

Uporabimo cilindrične koordinate:

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$\varphi\in[0,2\pi]$

$r\in[0,2] \Rightarrow z\in[r^2\sin^2\varphi-2,\ 2-r^2\cos^2\varphi]$

$$V=\int_{\varphi=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2\sin^2\varphi-2}^{2-r^2\cos^2\varphi} 1\cdot r\,dz\,dr\,d\varphi=$$

$$=\int_{\varphi=0}^{2\pi}\int_{r=0}^{2} r\cdot z\Big|_{r^2\sin^2\varphi-2}^{2-r^2\cos^2\varphi}dr\,d\varphi=$$

$$=\int_{\varphi=0}^{2\pi}\int_{r=0}^{2} r(2-r^2\cos^2\varphi-r^2\sin^2\varphi+2)\,dr\,d\varphi=$$

$$=\int_{\varphi=0}^{2\pi}\int_{r=0}^{2} r(4-r^2)\,dr\,d\varphi=2\pi\int_0^2(4r-r^3)\,dr=$$

$$=2\pi\left(2r^2-\frac{r^4}{4}\right)\Big|_0^2=2\pi\left(8-\frac{16}{4}\right)=8\pi$$

$$m=\int_{\varphi=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2\sin^2\varphi-2}^{2-r^2\cos^2\varphi} r^3\sin\varphi\,dz\,dr\,d\varphi= \ldots$$ sami doma :)