

Kroneckerjev produkt

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1. KRONECKERJEV (TENZORSKI) PRODUKT

Kroneckerjev produkt (tudi **tenzorski produkt**) matrik $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ in $B \in \mathbb{R}^{p \times q}$ je $mp \times nq$ matrika

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp \times nq}.$$

Če so matrike A, B, C in D primernih velikosti, potem veljajo naslednje enakosti.

- (1) $0 \otimes A = A \otimes 0 = 0$
- (2) $\alpha \otimes A = A \otimes \alpha = \alpha A$ za vsak $\alpha \in \mathbb{R}$
- (3) $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$
- (4) $(A + B) \otimes C = A \otimes C + B \otimes C$ in $A \otimes (B + C) = A \otimes B + A \otimes C$
- (5) $(A \otimes B)^T = A^T \otimes B^T$
- (6) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
- (7) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
- (8) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
- (9) $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$
- (10) $\text{rang}(A \otimes B) = \text{rang}(A) \text{rang}(B)$
- (11) Če ima matrika $A \in \mathbb{R}^{n \times n}$ lastne vrednosti $\lambda_1, \dots, \lambda_m$ in ima matrika B lastne vrednosti μ_1, \dots, μ_n , potem je množica lastnih vrednosti matrike $A \otimes B$ enaka

$$\{\lambda_i \mu_j; \lambda_i \text{ lastna vrednost } A, \mu_j \text{ lastna vrednost } B\}.$$
- (12) Če $A \in \mathbb{R}^{n \times n}$ in $B \in \mathbb{R}^{m \times m}$, potem je $\det(A \otimes B) = (\det A)^m (\det B)^n$.

2. PRESLIKAVA vec

Za matriko $A \in \mathbb{R}^{m \times n}$ označimo **vektorizacijo matrike A** kot

$$\text{vec}(A) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(n)} \end{bmatrix} \in \mathbb{R}^{mn}.$$

vec je preslikava iz $\mathbb{R}^{m \times n}$ v \mathbb{R}^{mn} .

Izrek 1. Za matrike $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ in $C \in \mathbb{R}^{p \times r}$ velja

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B).$$

3. NADALJNJE BRANJE

- (1) David A. Harville: Matrix Algebra From a Statistian's Perspective, Springer, 1997, razdelki 16.1., 16.2. in 16.3.
- (2) Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis*, Cambridge, 2006, razdelek 4.2.
- (3) Charles F. Van Loan: The ubiquitous Kronecker product, Journal of Computational and Applied Mathematics 123 (2000) 85-100.

4. DOMAČA NALOGA

- (1) Rešite kviz na spletni Učilnici.
- (2) David A. Harville: Matrix Algebra From a Statistian's Perspective, Springer, 1997, stran 368 (Exercises 1, 2, 3, 4, 14, 15, 16).