Problem:

$\boxed{P}$ $f: \mathbb{R}^n \to \mathbb{R}$

$$\min f(\underline{x})$$

pri pogojih $g_i(\underline{x}) = 0 \quad i = 1, \ldots, m$

$h_i(\underline{x}) \leq 0 \quad j = 1, \ldots, p$

Lani: $p = 0$

$$\vec{G}(\underline{x}) = \begin{bmatrix} g_1(\underline{x}) \\ \vdots \\ g_m(\underline{x}) \end{bmatrix} \qquad \vec{G}: \mathbb{R}^n \to \mathbb{R}^m$$

$[\lambda_1, \ldots, \lambda_m \quad \mu_1, \ldots, \mu_p]$ ← prirejene spremenljivke

$$L(\underline{x}, \underline{\lambda}, \underline{\mu}) = f(\underline{x}) - \underline{\lambda} \cdot \vec{G}(\underline{x}) - \underline{\mu} H(\underline{x})$$

Lagrangeova funkcija problema P

$$= f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \ldots - \lambda_m g_m(\underline{x}) - \mu_1 h_1(\underline{x}) - \ldots - \mu_p h_p(\underline{x})$$

Letos: p poljuben

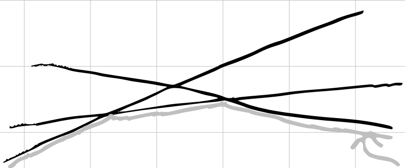
$$\vec{H}(\underline{x}) = \begin{bmatrix} h_1(\underline{x}) \\ \vdots \\ h_p(\underline{x}) \end{bmatrix} \qquad \vec{H}: \mathbb{R}^n \to \mathbb{R}^p$$

Torej: $L: \mathbb{R}^{n+m+p} \to \mathbb{R}$

$$K(\underline{\lambda}, \underline{\mu}) = \underset{\underline{x} \in D}{\inf} L(\underline{\lambda}, \underline{\mu}) =$$

najmanjša vrednost

$$= \inf_{\underline{x}} \{ \underbrace{f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \ldots - \lambda_m g_m(\underline{x}) - \mu_1 h_1(\underline{x}) - \ldots - \mu_p h_p(\underline{x})}_{\text{vsaka linearna v } \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_p} \}$$



lomljena linearna funkcija

prirejena funkcija problemu $\boxed{P}$

$K(\underline{\lambda}, \underline{\mu})$ konkavna

Naj bo $\underline{x}^*$ tista vrednost $\underline{x}$, ki reši problem $\boxed{P}$.

$f(\underline{x}^*) \leq f(\underline{x})$ za $\forall \underline{x}$, ki zadošča pogojem:

$p^* = \min f(\underline{x})$, ki zadoščajo pogojem $g_i(\underline{x}) = 0$ in $h_j(\underline{x}) \leq 0$.

Ker $\underline{x}^*$ zadošča robnim pogojem $\boxed{P}$:

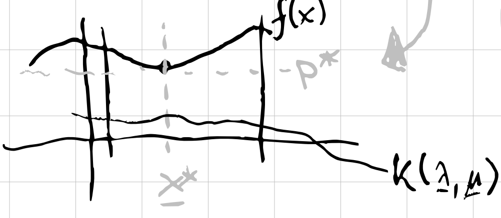
$g_i(\underline{x}^*) = 0$ in $h_j(\underline{x}^*) \leq 0$.

$$K(\underline{\lambda}, \underline{\mu}) = \inf_{\underline{x}} \{ f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \ldots - \lambda_m g(\underline{x}) - \mu_1 h_1(\underline{x}) - \ldots - \mu_p h_p(\underline{x}) \} \leq$$

$$\leq f(\underline{x}^*) - \lambda_1 \underbrace{g_1(\underline{x}^*)}_{= 0} - \ldots - \lambda_m \underbrace{g(\underline{x}^*)}_{= 0} - \mu_1 \underbrace{h_1(\underline{x}^*)}_{\leq 0} - \ldots - \mu_p \underbrace{h_p(\underline{x}^*)}_{\leq 0} \leq$$

če $\mu_1, \ldots, \mu_p \leq 0$

$$\leq f(\underline{x}^*) = p^*$$



$$\boxed{\begin{gathered} K(\underline{\lambda}, \underline{\mu}) \leq p^* \\ \text{za } \forall \lambda \text{ in za } \forall \mu \leq 0 \end{gathered}}$$

Iz tega sledi:

$$\max_{\substack{\underline{\lambda} \\ \mu_1, \dots, \mu_p \leq 0}} K(\underline{\lambda}, \underline{\mu}) \leq p^*$$

$$\Updownarrow$$

$\boxed{P}$ $\min f(\underline{x})$

p. pogojih $g_i(\underline{x}) = 0 \quad i = 1, \dots, m$

$h_j(\underline{x}) = 0 \quad j = 1, \dots, p$

rešitev: p^*

Originalni problem v spremenljivkah $\underline{x}$

Prirejeni problem v spremenljivkah $\underline{\lambda}, \underline{\mu}$

$$\Updownarrow$$

konkavna

$\boxed{D}$ $\max_{\underline{\lambda}, \underline{\mu}} K(\underline{\lambda}, \underline{\mu})$

konveksni

p. pogojih $\mu_1, \dots, \mu_p \leq 0$

je konveksen problem $\Rightarrow$ obstajajo učinkoviti algoritmi

Naj $\underline{\lambda}^*, \underline{\mu}^*$ rešita $\boxed{D}$ in $d^* = \max_{\substack{\underline{\lambda}, \underline{\mu} \\ \mu_j \leq 0}} K(\underline{\lambda}, \underline{\mu}) = K(\underline{\lambda}^*, \underline{\mu}^*)$

Vemo: $d^* \leq p^*$

Vprašanja:

- Kdaj je $d^* = p^*$?
- Koliko je lahko prirejena vrzel $p^* - d^*$?

Primer: Linearno programiranje (vrzel je 0: $d^* = p^*$)

$$\min_{\underline{x}} \underline{c} \cdot \underline{x}^T \left(= c_1 x_1 + c_2 x_2 + \dots\right)$$

pri pogojih $A\underline{x}^T \leq \vec{b}$ (m pogojev)

po komponentah

Lagrangeva funkcija:

$$L(\underline{x}, \underset{\in \mathbb{R}^p}{\underline{\mu}}) = \underline{c} \cdot \underline{x}^T - \underline{\mu}(A\underline{x}^T - \vec{b})$$

Prirejena funkcija:

$$K(\underline{\mu}) = \inf_{\underline{x}} \{\underline{c}\underline{x}^T - \underline{\mu}(A\underline{x}^T - \vec{b})\} = \underline{\mu}\vec{b} + \inf_{\underline{x}} \{(\underline{c} - \underline{\mu}A)\underline{x}^T)\} =$$

NEODV. OD $\underline{x}$

$$= \begin{cases} \underline{\mu}\vec{b} \; ; & \text{ko } \underline{c} = \underline{\mu}A \\ -\infty \; ; & \text{sicer} \end{cases}$$

$$\max_{\mu_1, \dots, \mu_p \leq 0} K(\underline{\mu}) \implies$$

$\boxed{D}$ $\max \underline{\mu}\vec{b}$

p. pogojih $\mu_1 \leq 0$
$\vdots$
$\mu_p \leq 0$

Kaj lahko povemo o $\underline{x}^*, \underline{\lambda}^*, \underline{\mu}^*$, ko $d^* = p^*$?

$$d^* = K(\underline{\lambda}^*, \underline{\mu}^*) = \inf_{\underline{x}} \Big\{ f(\underline{x}) - \sum_{i=1}^{m} \lambda_i^* g_i(\underline{x}) - \sum_{j=1}^{p} \mu_j^* h_j(\underline{x}) \Big\} \le$$

$$\overset{1}{\le} f(\underline{x}^*) - 0 - \dots - 0 - \underbrace{\mu_1^*}_{\le 0} \underbrace{h_1(\underline{x}^*)}_{\le 0} - \dots - \underbrace{\mu_p^*}_{\le 0} \underbrace{h_p(\underline{x}^*)}_{\le 0} \le$$

$$\overset{2}{\le} f(\underline{x}^*) = p^*$$

Takrat so tu enakosti!

Za $\boxed{1}$ velja enačaj:

$$\inf_{\underline{x}} L(\underline{x}, \underline{\lambda}^*, \underline{\mu}^*) = L(\underline{x}^*, \underline{\lambda}^*, \underline{\mu}^*)$$

Pogoj: $$\left. \frac{\partial L(\underline{x}, \underline{\lambda}^*, \underline{\mu}^*)}{\partial \underline{x}^T} \right|_{\underline{x} = \underline{x}^*} = 0$$

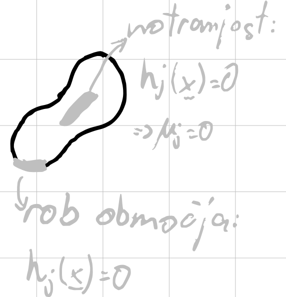
n enačb

Za $\boxed{2}$:

$$\underbrace{\mu_1^*}_{\le 0} \underbrace{h_1(\underline{x}^*)}_{\le 0} + \dots + \underbrace{\mu_p^*}_{\le 0} \underbrace{h_p(\underline{x}^*)}_{\le 0} = 0$$

Torej $\boxed{\mu_j^* h_j(\underline{x}^*) = 0}$ za $\forall j = 1, \dots, p$

prirejeno dopolnjevanje



Za $\forall j$: $\mu_j = 0$ ALI $h_j(\underline{x}^*) = 0$

In vsi robni pogoji $\boxed{3}$ $g_i(\underline{x}^*) = 0$, $h_j(\underline{x}^*) \le 0$

$\boxed{4}$ $\mu_j^* \le 0$ za $j = 1, \dots, p$

Pogojev za λ ni.

Imamo torej $n + p + m + p + p = n + m + 3p$ pogojev. ($\boxed{1}$ do $\boxed{4}$)

Imenujejo se Karush-Kuhn-Tuckerjevi pogoji problema $\boxed{P}$.

Pokazali smo, da v primeru $p^* = d^*$, so rešitve $\underline{x}^*$ problema $\boxed{P}$ tudi rešitve KKT pogojev.

<u>Primer:</u> Zapišimo KKT pogoje problema:

$$\min \ (x-2)^2 + 2(y-1)^2$$

pri pogojih $\quad x + 4y \leq 3$

$\qquad\qquad\qquad x \geq y$

pazi!

$$L(x, y, \mu_1, \mu_2) = (x-2)^2 + 2(y-1)^2 - \mu_1(x+4y-3) - \mu_2(y-x)$$

$\boxed{1}$ Parcialna odvoda po orig. spr. morata biti 0.

$$\frac{\partial L}{\partial x} = 2(x-2) - \mu_1 + \mu_2 = 0$$

$$\frac{\partial L}{\partial y} = 4(y-1) - 4\mu_1 - \mu_2 = 0$$

h $\boxed{2}$ Prirejeni pogoji enaki 0. (samo za neenakosti!) μ

$$\mu_1(x+4y-3) = 0$$

$$\mu_2(y-x) = 0$$

hing $\underline{\boxed{3}}$ $x + 4y \leq 3$ in $x \geq y$ (tudi za enakosti) λ

h $\boxed{4}$ $\mu_1 \leq 0 \qquad \mu_2 \leq 0$