3.4.1 Optimizacija

(funkcij več spremenljivk pri dodatnih pogojih)

$f : \mathbb{R}^n \to \mathbb{R}$

Problem: Poiščimo max/min $f(\underline{x})$

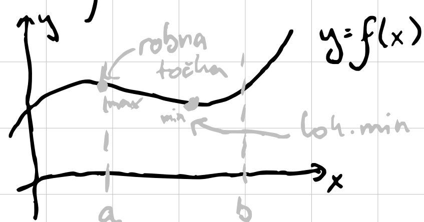
pri pogojih

$g_i(\underline{x}) = 0 \qquad i = 1, \dots, m$

$h_j(\underline{x}) \leq 0 \qquad j = 1, \dots, p$

Najpreprostejši primer:

① $n=1$: $f : \mathbb{R} \to \mathbb{R}$ $\qquad m=0 \quad p=2$



max/min $f(x)$

pri pogoju: $x - b \leq 0$

$a - x \leq 0$

Torej $x \leq b$ in $x \geq a \Rightarrow a \leq x \leq b$

② Danes: $n=2$ $\qquad$ min/max $f(x,y)$

$m=1$

$p=0$ $\qquad$ pri pogoju $g(x,y)=0$

③ Poljubni m, n, p

f, g, h linearne funkcije

($\max/\min \; A\vec{x}$

pri pogoju $C\vec{x} = \vec{d}$

$\vec{x} \leq 0$)

Temu se reče linearno programiranje.

④ f, h konveksni → konveksni optimizacijski problem

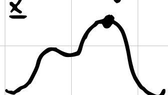
Problem: $\min f$

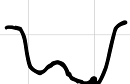
pri pogojih $g_i(\underline{x}) = 0 \quad i = 1, \dots, m$

$h_j(\underline{x}) \leq 0 \quad j = 1, \dots, p$

Preureditve:

① $\max\limits_{\underline{x}} f \leftrightarrow \min\limits_{\underline{x}} -f$





② $A \geq B \leftrightarrow B - A \leq 0$

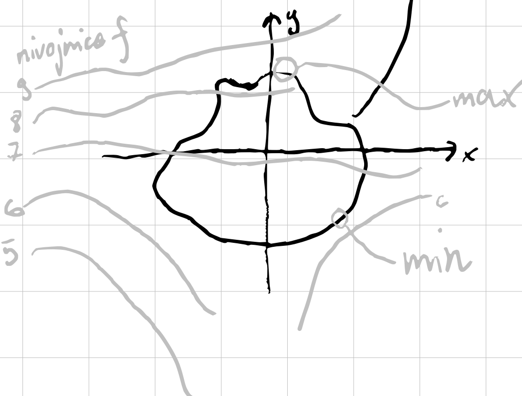
③ $A \leq B \leftrightarrow B = A + t$

t .. vpeta spr.

④ $A = B \leftrightarrow A \leq B$ in $B \leq A$

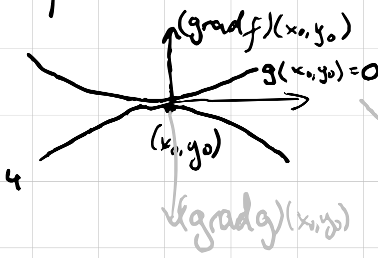
1. posebni primer: $f: \mathbb{R}^2 \to \mathbb{R}$

Iščemo min f
pri pogoju $g(x, y) = 0$ ← implicitno podana krivulja



Nivojske krivulje

$g(x, y) = 0$ in nivojnica f sta si v točki, kjer f doseže minimum na $g(x,y)=0$, tangentni.



(x_0, y_0) leži na $g=0$

→ implicitno dana krivulja

$g(x,y) = 0 \quad / \frac{\partial}{\partial t} \quad \to x = x(t), \ y = y(t)$

$$\frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} = 0$$

$$\underbrace{\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right]}_{\text{grad } g} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = 0$$

$(\operatorname{grad} f)(x_0, y_0)$ in
$(\operatorname{grad} g)(x_0, y_0)$
sta vzporedna

$$\Rightarrow (\operatorname{grad} f)(x_0, y_0) = \underbrace{\lambda}_{\text{Lagrangeev multiplikator}} (\operatorname{grad} g)(x_0, y_0)$$

$$L(x, y, \lambda) = f(x,y) - \lambda g(x,y)$$ Lagrangeva funkcija

Kaj so stacionarne točke funkcije L?

$$\left.\begin{aligned} \bullet\ \frac{\partial L}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \bullet\ \frac{\partial L}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \end{aligned}\right\} \text{v } (x_0, y_0) \text{ P je } \frac{\partial L}{\partial x}(x_0,y_0) = 0, \quad \frac{\partial L}{\partial y}(x_0,y_0) = 0$$

P

$$\xrightarrow[\text{prejš. strani}]{\text{pogoj na}} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(x_0, y_0) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

$$\bullet\ \frac{\partial L}{\partial \lambda} = -g(x,y)$$

$$\frac{\partial L}{\partial \lambda}(x_0, y_0) = 0 \quad (\text{ker } (x_0, y_0) \in g(x,y) = 0)$$

Če (x_0, y_0) reši problem na prejšnji strani
(min f pri pogoju $g = 0$),
potem je (x_0, y_0, λ_0) stac. točka $L(x,y,\lambda)$ za
nek λ_0.

Zato poiščemo rešitev problema tako:

1. $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ zapišemo enačbo
2. Poiščemo stac. točke $S_1, \dots, S_R$ funkcije L

   $S_1 = (x_1, y_1, \lambda_1)$
3. Izračunamo $f(x_1, y_1), \dots, f(x_R, y_R)$
4. Izberemo najmanjšo vrednost

Primer:

V katerih točkah na območju, ki ga opisuje neenačba

$$4(x-1)^2+y^2 \leq 16,$$

zavzame funkcija

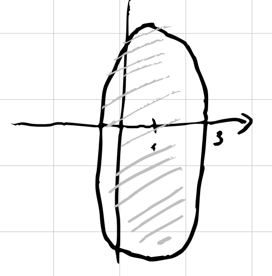
$$f(x,y)=2x^2+y^2$$

najmanjšo in največjo vrednost?

Rob območja:

$$4(x-1)^2+y^2=16$$

$$\frac{(x-1)^2}{2^2}+\frac{y^2}{4^2}=1 \quad \text{(elipsa)}$$



1\. primer: točke v notranjosti so lok. ekstremi

Iščemo stac. točke funkcije f.

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=4x\\ \frac{\partial f}{\partial y}&=2y\end{aligned}\right\} \begin{aligned}4x&=0\\2y&=0\end{aligned} \Rightarrow x=y=0 \quad S_0(0,0)$$

$$\boxed{f(S_0)=0}$$

2\. primer: točke na robu

min/max $f(x)$

pri pogoju $4(x-1)^2+y^2-16=0$

1 $L(x,y,\lambda)=2x^2+y^2-\lambda(4(x-1)^2+y^2-16)=0$

2 Poiščemo stac. točke L

$$\frac{\partial L}{\partial x}=4x-8\lambda x+8\lambda$$

$$\frac{\partial L}{\partial y}=2y(1-\lambda)$$

$$\frac{\partial L}{\partial \lambda}=-(4(x-1)^2+y^2-16) \quad (-g)$$

$\frac{\partial L}{\partial y}=0 \Rightarrow$ ① $y=0 \Rightarrow 4(x-1)^2-16 \Rightarrow x=3$ ali $x=-1$

$S_1(3,0) \quad S_2(-1,0)$

② $y\neq 0 \Rightarrow \lambda=1 \Rightarrow 4x-8x+8=0 \Rightarrow x=2$

$=4\cdot 1+y^2=16 \Rightarrow y^2=12 \Rightarrow y=\pm 2\sqrt{3}$

$S_3(2,2\sqrt{3}) \quad S_4(2,-2\sqrt{3})$

3 Poračunamo vse $f(S)$

$$f(3,0)=18$$

$$f(-1,0)=2$$

$$f(S_3)=f(S_4)=20$$

Minimum f na območju je 0 (v točki $(0,0)$)

Maksimum f na območju je 20 (v $(2,\pm 2\sqrt{3})$)

Bolj splošno:

$$\left.\begin{array}{l} \min\ f(x_1,\dots,x_n) \\ \text{pri pogojih}\quad g_1(x_1,\dots,x_n)=0 \\ \qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad g_m(x_1,\dots,x_n)=0 \qquad (p=0) \end{array}\right] \text{P}$$

poenostavljeno:

$\min f(\underline{x})$

pri pogoju $\quad \vec{G}(\underline{x})=0, \quad \vec{G}:\mathbb{R}^n \to \mathbb{R}^m$

$$\vec{G}(\underline{x}) = \begin{bmatrix} g_1(\underline{x}) \\ \vdots \\ g_m(\underline{x}) \end{bmatrix}$$

Lagrangeva funkcija

$$L(\underbrace{x_1,\dots,x_n}_{n},\underbrace{\lambda_1,\dots,\lambda_m}_{m}) = f(x_1,\dots,x_n) - \lambda_1 g_1(x_1,\dots,x_n) - \dots - \lambda_m g_m(x_1,\dots,x_n)$$

$n+m$ spr.

$$L(\underline{x},\underline{\lambda}) = f(\underline{x}) - \underline{\lambda}\,\vec{G}(\underline{x})$$

Točke $\underline{x}$, ki rešijo P, so stac. točke $L(\underline{x},\underline{\lambda})$

(če $(\operatorname{grad} g_1)(\underline{x}),\dots,(\operatorname{grad} g_m)(\underline{x})$ lin. neodvisni)

Primer: $A \in \mathbb{R}^{n\times n}$ simetrična

Poiščimo največjo vrednost $\frac{\vec{x}^T A \vec{x}}{\vec{x}^T\vec{x}}$ $(\vec{x} \in \mathbb{R}^n, \vec{x} \neq 0)$.

$$\alpha\vec{x}: \frac{(\alpha\vec{x})^T A(\alpha\vec{x})}{(\alpha\vec{x})^T(\alpha\vec{x})} = \frac{\alpha^2\vec{x}^T A\vec{x}}{\alpha^2\vec{x}^T\vec{x}} = \frac{\vec{x}^T A\vec{x}}{\vec{x}^T\vec{x}}$$

Dovolj je preveriti en vektor v vsako smer.

Ekvivalentno:

Poiščimo max izraza $\frac{x^T A x}{x^T x}$ $(\vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1)$

$\iff$ iščemo max $x^T A x$ $(\vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1)$

$\|\vec{x}\|^2 - 1 = 0$

(1 en vezni pogoj)

1 v.p., 1 λ

$$L(\vec{x}, \lambda) = \vec{x}^T A\vec{x} - \lambda(\|\vec{x}\| - 1)$$

$$0 = \frac{\partial L}{\partial \vec{x}} = 2\vec{x}^T A - 2\lambda\vec{x}^T$$ (napiši na list za kolokvij)

$$0 = \frac{\partial L}{\partial \lambda} = -(\|\vec{x}\|^2 - 1)$$

Rešimo:

$$2A\vec{x} - 2\lambda\vec{x} = \vec{0} \quad /:2$$

$$\|\vec{x}\|^2 = 1$$

$$A\vec{x} = \lambda\vec{x}$$

Rešitve $\lambda_1, \ldots, \lambda_n$: last. vr. matr. A

$x_1, \ldots, x_n$: last. vekt. dolžine 1, ki pripadajo $\lambda_1, \ldots, \lambda_n$

$$\max_{\substack{\text{l. vekt. } A \\ \text{dolž. } 1}} \vec{x}_i^T \underbrace{A\vec{x}_i}_{\lambda_i x_i} = \max_{-\|-} \lambda_i \|\vec{x}_i\|^2 = \max \lambda_i$$

Odgovor: največja taka vrednost je največja lastna vrednost matrike A.