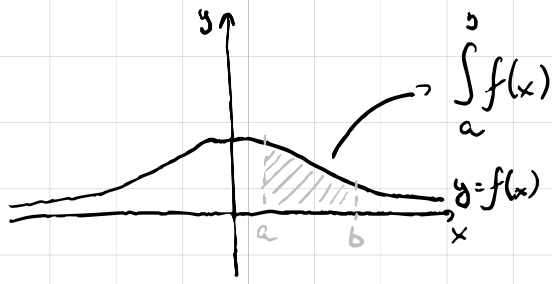
$F: \mathbb{R}^n \to \mathbb{R}^n$ vektorske funkcije več spremenljivk

$\boxed{3.2.}$ Večkratni integral (funkcije več spremenljivk)

$f: \mathbb{R}^n \to \mathbb{R}^1$

Za $n=1$:



$\int_a^b f(x)\,dx$, če $f \geq 0$

Če je interval $[a,b]$ sestavljen $\to \int_a^b \cong \sum$ površina pravokotnika

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{\frac{b-a}{n}}_{\text{širina}} \cdot \underbrace{f\left(a+j\frac{b-a}{n}\right)}_{\text{višina}}$$

Za poljuben n definiramo podobno:

$n=2$: $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$

$$\iint_D f(x,y)\,dx\,dy = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum\sum \underbrace{\Delta x \Delta y}_{\text{ploščina osn. ploskve kvadra}} \underbrace{f(x^*, y^*)}_{\text{višina kvadra}}$$

točka v osnovni ploskvi

dvojni integral

$f \geq 0$ na D, potem je $\iint_D f(x,y)\,dx\,dy$ prostornina telesa na D omejen z grafom $z = f(x,y)$.

Primer.

$K = \{(x,y);\ x^2+y^2 \leq 1\}$

$\iint_K \sqrt{1-x^2-y^2}\,dx\,dy$

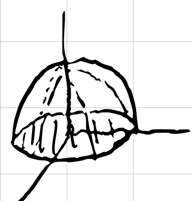
definicijsko območje

$z = \sqrt{1-x^2-y^2} \geq 0$

$z^2 = 1-x^2-y^2$

$x^2+y^2+z^2 = 1$ krogla!

kupola



$\iint_K = \frac{2}{3}\pi$ (polovica prostornine krogle)

$\iint_D dx\,dy$ $\int_{D_y}\left(\int_{D_x} dx\right)dy$

Kako pretvorimo dvojni integral na dvakratnega?

<u>Izrek (Fubini)</u>

$\boxed{1}$ $D = [a,b]\times[c,d]$ pravokotnik, potem

$$\iint_D f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

narežemo na rezine  

Primer: Integrirajmo $f(x,y)=xy+e^{2y}$ na $\mathcal{D}=\{(x,y);\ 1\le x\le 2,\ 0\le y\le 3\}$.

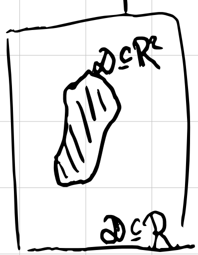
$$\iint_{\mathcal{D}}(xy+e^{2y})\,dx\,dy=\int_0^3\left(\int_1^2(xy+e^{2y})\,dx\right)dy=$$

$$=\int_0^3\left(\frac{x^2}{2}y+e^{2y}x\right)\Big|_{x=1}^{2}dy=\int_0^3\left(\frac{4}{2}y+2e^{2y}-\frac{1}{2}y-e^{2y}\right)dy=$$

$$=\frac{3}{2}\int_0^3 y\,dy+\int_0^3 e^{2y}\,dy=\frac{3}{2}\,\frac{y^2}{2}\Big|_{y=0}^{3}+\frac{1}{2}e^{2y}\Big|_{y=0}^{3}=$$

$$=\frac{27}{4}+\frac{1}{2}e^6-\frac{1}{2}=\frac{25}{4}+\frac{1}{2}e^6$$

$\boxed{2}$ $\mathcal{D}$ ni pravokotnik... $\iint_{\mathcal{D}} f(x,y)\,dx\,dy=?$



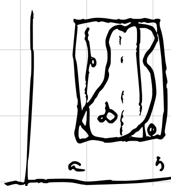
$f:\mathcal{D}\to\mathbb{R}$

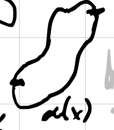
$\mathcal{D}\subseteq$ pravokotnik $\mathcal{R}\subseteq\mathbb{R}^2$

$F:\mathcal{R}\to\mathbb{R}$

da velja $\iint_{\mathcal{R}} F\,dx\,dy=\iint_{\mathcal{D}} f\,dx\,dy$

$$F(x,y)=\begin{cases} f(x,y); & (x,y)\in\mathcal{D}\\ 0; & (x,y)\in\mathcal{R}\setminus\mathcal{D}\end{cases}$$





$$\iint_{\mathcal{D}} f(x,y)\,dx\,dy=\int_a^b\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right)dx$$

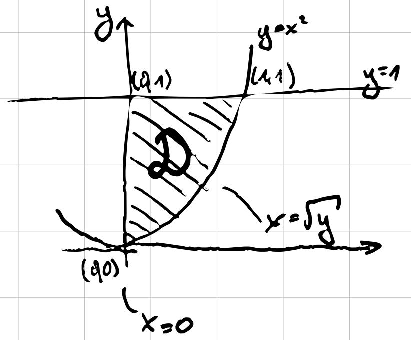
Imamo zgornjo in spodnjo mejo območja

Ali vodoravno...

$$=\int_c^d\left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\right)dy$$

↑ zunanja meja vedno število, notri pa je funkcija

Primer: $\iint\limits_{D} \frac{x}{y}\,dx\,dy$, kjer je D območje v prvem kvadrantu nad $y=x^2$ in pod $y=1$.



meja za y

odvisna

$$\iint\limits_{D} \frac{x}{y}\,dx\,dy = \int_0^1 \left(\int_0^{\sqrt{y}} \frac{x}{y}\,dx \right) dy$$

meja za x

tudi $= \int_0^1 \left(\int_{x^2}^1 \frac{x}{y}\,dy \right) dx =$

$$= \int_0^1 \left(x \ln|y| \right)\Big|_{y=x^2}^{1} dx = \int_0^1 \left(-2x\ln(x) \right) dx =$$

$$= -2\int_0^1 x\ln(x)\,dx = -2\left(\frac{x^2}{2}\ln(x)\Big|_{x=0}^{1} - \int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx \right) =$$

per partes

$u=\ln(x)$ $\quad du = \frac{1}{x}dx$ $\quad x\,dx = dv$ $\quad \frac{x^2}{2} = v$

$$= -2\left(0 - 0 - \frac{1}{2}\int_0^1 x\,dx\right) = +\int_0^1 x\,dx = +\frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

L'Hospital: $\frac{1}{2}\lim\limits_{x\to 0} \underbrace{x^2}_{\to 0}\underbrace{\ln(x)}_{\to -\infty} = \frac{1}{2}\lim\limits_{x\to 0}\frac{\ln(x)}{x^{-2}} = \frac{1}{2}\lim\limits_{x\to 0}\frac{\frac{1}{x}}{-2x^{-3}} = -\frac{1}{4}\lim\limits_{x\to 0} x^2 = 0$

Primer: Kaj je $\iint\limits_{D} dx\,dy$?

$$\iint\limits_{D} dx\,dy = \int_0^1 \left(\int_0^{\sqrt{y}} dx \right) dy = \int_0^1 (\sqrt{y} - 0)\,dy$$

ploščina

$\left|\iint\limits_{D} dx\,dy\right|$ ploščina D

Podobno prostornina območja V v $\mathbb{R}^3$:

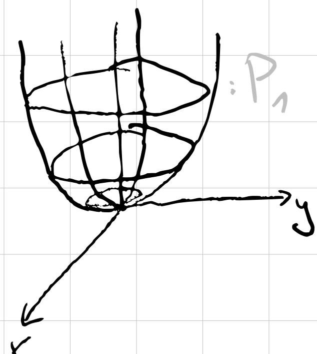
$$\left|\iiint\limits_{V} dx\,dy\,dz\right|$$

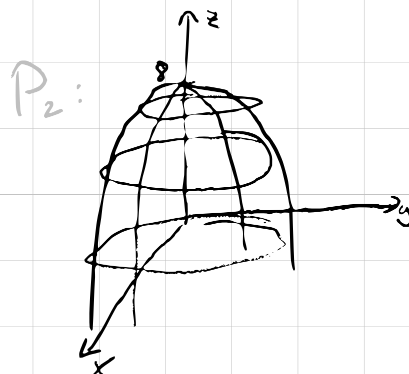
$\boxed{3}$ Menjava spremenljivk

<u>Primer</u>: Dana sta paraboloida

P_1 $z = x^2+y^2$ in P_2 $z = 8-(x^2+y^2)$.

Naj bo D omejeno območje med njima.
Izračunajmo prostornino D.





$$V = \left|\iiint_D dx\,dy\,dz\right| = \int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_{x^2+y^2}^{8-(x^2+y^2)} dz\right)dy\right)dx$$

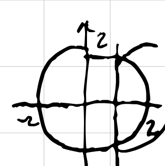
① Kje se P_1 in P_2 sekata?

$$\left.\begin{aligned} z &= x^2+y^2 \\ z &= 8-(x^2+y^2) \end{aligned}\right\} \Rightarrow x^2+y^2 = 8-(x^2+y^2)$$

$x^2+y^2=4$ krožnica z radijem 2

$z=4$ višina

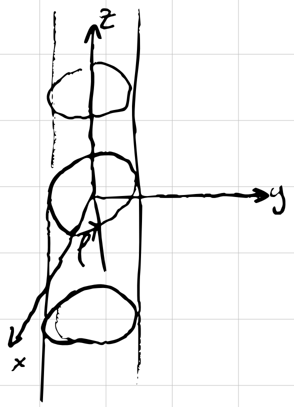
Pri fiksnem x:



$y = \pm\sqrt{4-x^2}$

V je težko izračunati... kaj pa v drugih koord. sistemih?

Cilindrične/valjne koordinate



$x = r\cos\varphi$

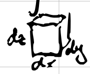
$y = r\sin\varphi$

$z = z$

$|J_{CILINDRIČNE}| = r$

$= |J_{POLARNE}|$

diferencial



torta, njam

kartezične $\longleftrightarrow$ cilindrične

$n=1 \quad \int f(x)\,dx = \int f(\Psi(u))\,\Psi'(u)\,du$

$u = \varphi(x) \longrightarrow x = \Psi(u)$

$du = \varphi(x)dx \rightarrow dx = \Psi(u)du$

$$\iint f(x,y)\,dx\,dy = \iint f(\Psi(u,v), \xi(u,v)) \cdot |\det J_{(x,y)\to(u,v)}|\,du\,dv$$

$x = \Psi(u,v)$

$y = \xi(u,v)$

Jacobijeva matrika!

$|J_{(x,y)\to(u,v)}|$

od prej...

$$= \int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_{x^2+y^2}^{8-(x^2+y^2)} dz\right)dy\right)dx =$$

$$= \int_{0}^{2\pi}\left(\int_{0}^{2}\left(\int_{r^2}^{8-r^2} r\,dz\right)dr\right)d\varphi$$

$\det(J_{CIL})$

$x^2+y^2 = r^2(\cos^2\varphi + \sin^2\varphi) = r$

$P_1: \ z = r^2$

$P_2: \ z = 8 - r^2$

$\varphi \in [0, 2\pi]$

Kje se sekata?

$r^2 = 8 - r^2 \Rightarrow r = 2 \quad (r \geq 0)$

$z = 4$