$\boxed{3}$ Funkcije in vektorske f. več spremenljivk

$$F: \mathbb{R}^n \to \mathbb{R}^m$$

$$F(x_1, \dots, x_n) = F(\underline{x}) = F(\vec{x})$$

← ne bomo razlikovali med vekt. spremenljivk v vrstici ali stolpcu

$$\underline{x} = (x_1, \dots, x_n)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$F(\vec{x}) := \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix}, \text{ kjer je } f_i: \mathbb{R}^2 \to \mathbb{R}$$

↖ funkcije več spremenljivk $(m=1)$

↖ vektorska funkcija več spremenljivk

Zgledi:

$$F: \mathbb{R}^n \to \mathbb{R}^m$$

funkcije več spremenljivk

m \ n	1	2	3	...	n
1	$f: \mathbb{R} \to \mathbb{R}$	$f: \mathbb{R}^2 \to \mathbb{R}$	$f: \mathbb{R}^3 \to \mathbb{R}$	...	$f: \mathbb{R}^n \to \mathbb{R}$
2	$f: \mathbb{R} \to \mathbb{R}^2$	$f: \mathbb{R}^2 \to \mathbb{R}^2$	...		
3	$f: \mathbb{R} \to \mathbb{R}^3$	$\vdots$	$\ddots$		

↑ parametrizirane krivulje

Obravnavali bomo le primere, kjer bodo $f_i : \mathbb{R}^n \to \mathbb{R}$ dovoljkrat zvezno odvedljive funkcije.

$$(\operatorname{grad} f_i)(\vec{x}) = \left(\frac{\partial f_i}{\partial x_1}(\vec{x}), \frac{\partial f_i}{\partial x_2}(\vec{x}), \dots, \frac{\partial f_i}{\partial x_n}(\vec{x})\right)$$

↑ smer najhitrejšega naraščanja f_i v točki $\vec{x}$

Torej $F : \mathbb{R}^n \to \mathbb{R}^n$ je sestavljena iz m takih funkcij.

Jacobijeva matrika (:= "matrika prvih odvodov)

$$J_F(x) = \begin{bmatrix} (\operatorname{grad} f_1)(\vec{x}) \\ (\operatorname{grad} f_2)(\vec{x}) \\ \vdots \\ (\operatorname{grad} f_n)(\vec{x}) \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_1 & \dots & x_n \end{array} \\ \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix} \end{array} \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array} =$$

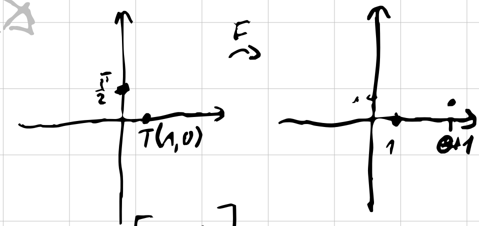
$$= \frac{\partial F}{\partial \vec{x}}$$ odvod funkcije F po vektorju $\vec{x}$ $$= \left[\frac{\partial f_i}{\partial x_j}\right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

Element Jacobijeve matrike v i-ti vrstici in j-tem st.

Primer:

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = \begin{bmatrix} e^x + \cos y \\ x + x\sin y \end{bmatrix}$$



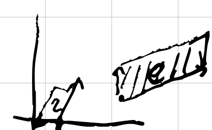
$$F(1,0) = \begin{bmatrix} e+1 \\ 1 \end{bmatrix}$$

$$F(0, \tfrac{\pi}{2}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Pa Jacobijeva matrika?

$$J_F(x,y) = \begin{array}{c c} & \begin{array}{cc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{array} \\ \begin{array}{c} \partial f_1 \\ \partial f_2 \end{array} & \begin{bmatrix} e^x & -\sin y \\ 1+\sin y & x\cos y \end{bmatrix} \end{array} = \frac{\partial F}{\partial (x,y)}$$

Kaj pa nam pove?



$$J_F(1,0) = \begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix}$$

za majhne spremembe je preslinearna 8

J. matr. nam pove spreminjanje F v MAJHNI OKOLICI.

$$J_F(0, \tfrac{\pi}{2}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$|\det J_F(\vec{x})|$ pove kvocient prostornine v okolici točke $F(\vec{x})$ in $\vec{x}$.

Denimo, da $F: \mathbb{R}^n \to \mathbb{R}^m$, $F(\vec{x}) = A\vec{x}$ za dano matriko $A \in \mathbb{R}^{m \times n}$.

$$F(\vec{x}) = A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} \begin{matrix} \text{"} f_1(x_1, \ldots, x_n) \\ \\ \text{"} f_m(x_1, \ldots, x_n) \end{matrix}$$

$$J_F(\vec{x}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A$$

torej $\dfrac{\partial F}{\partial \vec{x}} = \dfrac{\partial (Ax)}{\partial x} = A$

V posebnem:

$$\boxed{\frac{\partial (a^T x)}{\partial \vec{x}} = \vec{a}^T}$$

$$\frac{\partial (fg)}{\partial x} = \frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f$$

Kaj pa odvod produkta dveh funkcij?

$\vec{y} = \vec{y}(\vec{x}): \mathbb{R}^n \to \mathbb{R}^m$ $\qquad \vec{y}(\vec{x}) = \begin{bmatrix} y_1(\vec{x}) \\ \vdots \\ y_m(\vec{x}) \end{bmatrix}$ $\qquad y_i: \mathbb{R}^n \to \mathbb{R}$

$\vec{z} = \vec{z}(\vec{x}): \mathbb{R}^n \to \mathbb{R}^m$

$$\frac{\partial (\vec{y}(\vec{x})^T \vec{z}(\vec{x}))}{\partial \vec{x}} = \frac{\partial (\vec{y}_1(\vec{x}) z_1(\vec{x}) + \vec{y}_2(\vec{x}) z_2(\vec{x}) + \cdots + y_m(\vec{x}) z_m(\vec{x}))}{\partial \vec{x}}$$

$$= \frac{\partial}{\partial x}(y_1(\vec{x}) z_1(\vec{x})) + \cdots + \frac{\partial}{\partial x}(y_m(\vec{x}) z_m(\vec{x}))$$

i-ti stolpec J.m.

$$= \frac{\partial}{\partial x_i}(y_1(\vec{x}) z_1(\vec{x})) + \cdots + \frac{\partial}{\partial x_i}(y_m(\vec{x}) z_m(\vec{x})) =$$

$$= \frac{\partial y_1(\vec{x})}{\partial x_i} z_1(\vec{x}) + y_1(\vec{x}) \frac{\partial z_1(\vec{x})}{\partial x_i} + \cdots + \frac{\partial y_m(\vec{x})}{\partial x_i} z_m(\vec{x}) + y_m(\vec{x}) \frac{\partial z_m(\vec{x})}{\partial x_i}$$

$$= [z_1(\vec{x}) \ \ldots \ z_m(\vec{x})] \begin{bmatrix} \frac{\partial y_1(\vec{x})}{\partial \vec{x}} \\ \vdots \\ \frac{\partial y_m(\vec{x})}{\partial \vec{x}} \end{bmatrix} + [y_1(\vec{x}) \ \ldots \ y_m(\vec{x})] \begin{bmatrix} \frac{\partial z_1(\vec{x})}{\partial \vec{x}} \\ \vdots \\ \frac{\partial z_m(\vec{x})}{\partial \vec{x}} \end{bmatrix}$$

Sestavimo nazaj celo mat.

vrst stol.

$$\Rightarrow \frac{\partial (\vec{y}(\vec{x})^T \vec{z}(\vec{x}))}{\partial \vec{x}} = \underbrace{\vec{z}(\vec{x})^T}_{1 \times m} \underbrace{\frac{\partial \vec{y}(\vec{x})}{\partial \vec{x}}}_{m \times n} + \underbrace{\vec{y}(\vec{x})^T}_{1 \times m} \underbrace{\frac{\partial \vec{z}(\vec{x})}{\partial \vec{x}}}_{m \times n}$$

Primer:

$A \in \mathbb{R}^{n \times n}$

vrst. stol

$$\frac{\partial(\vec{x}^T A \vec{x})}{\partial \vec{x}} = (A\vec{x})^T \frac{\partial \vec{x}}{\partial \vec{x}} + \vec{x}^T \frac{\partial (A\vec{x})}{\partial \vec{x}} = \vec{x}^T A^T I + \vec{x}^T A = \vec{x}^T (A^T + A)$$

formula za odvod produkta $\vec{y}(\vec{x}) = \vec{x}$ $\vec{z}(\vec{x}) = A\vec{x}$

Kvadratno formo (f. veliko spremenljivk) odvajamo po veliko spremenljivkah

$$\boxed{\frac{\partial(\vec{x}^T A \vec{x})}{\partial \vec{x}} = \vec{x}^T (A^T + A)}$$

Če je A simetrična, je $A^T = A$

$$\frac{\partial(\vec{x}^T A \vec{x})}{\partial \vec{x}} = 2\vec{x}A$$

$n = 1$ $$\frac{\partial(xax)}{\partial x} = \frac{\partial(ax^2)}{\partial x} = 2ax$$

V posebnem: $$\boxed{\frac{\partial(\|\vec{x}\|^2)}{\partial \vec{x}} = 2\vec{x}^T}$$

$A = I$
odpoj

$$
\begin{tikzcd}[column sep=huge]
\mathbb{R}^n \arrow[r, bend left, "F"] & \mathbb{R}^m \arrow[r, bend left, "G"] & \mathbb{R}^\ell \\
\vec{x} \arrow[r, dashed, "\frac{\partial F}{\partial \vec{x}}", "m\times n"'] \arrow[rr, bend right=60, "\ell\times n", "\frac{\partial G}{\partial \vec{y}}\cdot\frac{\partial F}{\partial \vec{x}}"'] & F(\vec{x}) \arrow[r, dashed, "\frac{\partial G}{\partial \vec{y}}", "\ell\times m"'] & G(F(\vec{x})) \\
 & & (G\circ F)(\vec{x})
\end{tikzcd}
$$

$$\frac{\partial(G(F(\vec{x})))}{\partial \vec{x}} = \frac{\partial G(F(\vec{x}))}{\partial F(\vec{x})} \frac{\partial(F(\vec{x}))}{\partial \vec{x}}$$

$$\frac{\partial(e^{x^2})}{\partial x} = e^{x^2}\cdot 2x$$

$$x \mapsto x^2 \mapsto e^{x^2}$$

- $$\frac{\partial}{\partial \vec{x}} \|5\vec{x}\|^2 = 2(5\vec{x})^T \underbrace{\frac{\partial(5\vec{x})}{\partial \vec{x}}}_{5I} = 50\,\vec{x}^T$$

$$x \mapsto 5\vec{x} \mapsto \|5x\|^2$$

- A simetrična matrika

$$\frac{\partial}{\partial \vec{x}}\left(e^{-\frac{1}{2}\vec{x}^T A \vec{x}}\right) = e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \cdot \frac{\partial}{\partial \vec{x}}\left(-\frac{1}{2}\vec{x}^T A \vec{x}\right) = -\frac{1}{2} e^{-\frac{1}{2}x^T A x} \cdot 2x^T A = e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \cdot \vec{x}^T A$$