Vemo:

Simetrične matrike imajo vse lastne vrednosti realne.

A simetrična, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ l.vr.

$A = QDQ^T \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad Q = \begin{bmatrix} Q^{(1)} \dots Q^{(2)} \end{bmatrix}$

$AQ^{(i)} = \lambda Q^{(i)}$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A\vec{x} = n \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{bmatrix}$

linearna forma

$\vec{x}^T A \vec{x} = [x_1 \dots x_n] \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1(a_{11}x_1 + \dots + a_{1n}x_n) + \dots$

$+ x_n(a_{n1}x_1 + \dots + a_{nn}x_n) =$

$= a_{11}x_1^2 + \dots + a_{nn}x_n^2 +$

$(a_{12} + a_{21})x_1x_2 + (\quad)x_1x_3 + \dots + (\quad)x_{n-1}x_n$

kvadratna forma

Def: Za $A \in \mathbb{R}^{n \times n}$ imenujemo $\vec{x}^T A \vec{x}$ **kvadratna forma** matrike A.

Primer: $A = \begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{x}^T A \vec{x} = 5x_1^2 + 5x_2^2 + 2x_1x_2 + 0x_2x_1$

enako: $B = \begin{bmatrix} 5 & 0 \\ 2 & 5 \end{bmatrix}$ in $C = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

za vsako kv. formo obstaja natanko 1 simetrična matrika, ki ji ustreza

Def: Naj bo $A \in \mathbb{R}^{n\times n}$ simetrična. Pravimo, da je A...

1. Pozitivno semidefinitna, če je njena kvadratna forma nenegativna za vsak $\vec{x}$. $\forall \vec{x} \in \mathbb{R}^n . \vec{x}^T A \vec{x} \geq 0$.
2. Pozitivno definitna, če $\forall \vec{x} \in \mathbb{R}^n . \vec{x} \neq \vec{0} . \vec{x}^T A \vec{x} > 0$.
3. Negativno semidefinitna, če $\forall \vec{x} \in \mathbb{R}^n . \vec{x}^T A \vec{x} \leq 0$
4. Negativno definitna, če $\forall \vec{x} \in \mathbb{R}^n . \vec{x} \neq \vec{0} . \vec{x}^T A \vec{x} < 0$.
5. Nedefinitna, če nič od naštetega
   (tj. $\exists \vec{x} \in \mathbb{R} : \vec{x}^T A \vec{x} > 0$ in $\exists y \in \mathbb{R}^n : \vec{y}^T A \vec{y} < 0$)

linearna menjava spremenljivk

Kv.f. diag matr. so zelo lepe!

$\vec{u} = Q^T x$

$$A = QDQ^T \rightsquigarrow \underbrace{\vec{x}^T A \vec{x}}_{\text{kv. forma}} = \vec{x}^T Q \boxed{D} Q^T \vec{x} = \underbrace{\vec{u}^T D \vec{u}}_{\text{kv. forma v novih sprem.}}$$

kv. forma plus: v originalnih spremenljivkah

spremenili smo koord. sist.

kv. forma v novih sprem. $u_1 \ldots u_n$ plus: nima mešanih členov

diagonalizirali smo kv. formo

Primer: $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ $\quad \vec{x}^T A \vec{x} = 5x_1^2 + 5x_2^2 + 2x_1x_2$

Kaj predstavlja $5x_1^2 + 5x_2^2 + 2x_1x_2 = 24$?

Diagonalizirajmo A!

$\vec{u} = Q^T \vec{x}$

$\lambda_1 = 4, \quad \lambda_2 = 6 \quad \vec{x}^T A \vec{x} = \vec{u}^T D \vec{u} = 4\vec{u}_1^2 + 6\vec{u}_2^2$

$\rightsquigarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \rightsquigarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}_{Q^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

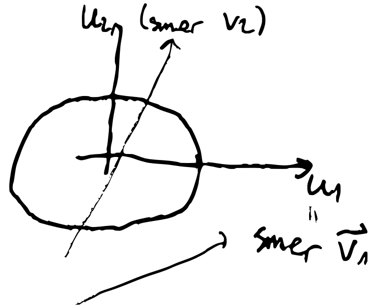
$u_1 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$

$u_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)$

Torej, prejšnja enačba se prepiše v

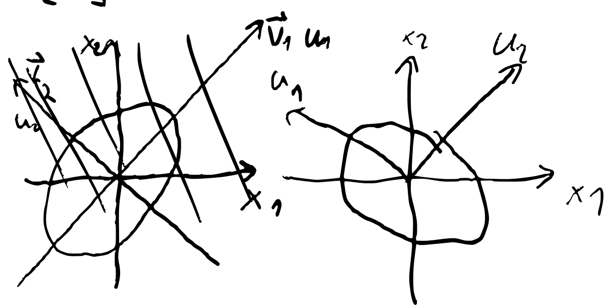
$4u_1^2+6u_2^2=24 \quad /:24$ (ampak v novih spremenljivkah!)

$$\frac{u_1^2}{(\sqrt{6})^2}+\frac{u_2^2}{2^2}=1$$



$$\vec{x}=Q\vec{u}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}Q^{(1)} & Q^{(2)}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$



Izrek:

$A\in\mathbb{R}^{n\times n}$ simetrična

① A je PSD natanko tedaj ko so vse lastne vrednosti matrike A nenegativne.

(dokaz: A PSD $\Rightarrow \vec{x}^TA\vec{x}\geq 0$ za vse $\vec{x}\in\mathbb{R}^n$

$\Rightarrow \vec{u}^TD\vec{u}\geq 0$ za vse $\vec{u}\in\mathbb{R}^n$

$\Rightarrow \lambda_1u_1^2+\lambda_2u_2^2+\ldots+\lambda_nu_n^2\geq 0$ za vse $\vec{u}\in\mathbb{R}^n$

$u_j=1 \quad u_0=u_1=\ldots=u_{j-1}=u_{j+1}=\ldots=u_n=0$

$\Rightarrow \lambda_j\geq 0$

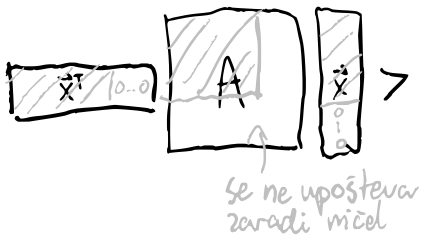
Če $\lambda_1 \ldots \lambda_n\geq 0 \Rightarrow \vec{x}^TA\vec{x}=\lambda_1\overset{\geq 0}{u_1^2}+\ldots+\lambda_n\overset{\geq 0}{u_n^2}\geq 0 \Rightarrow A$ PSD $\square$)

① A je PD $\Leftrightarrow$ $\lambda_1, \dots, \lambda_n > 0$.

③ A je NSD $\Leftrightarrow$ $\lambda_1 \dots \lambda_n \leq 0$

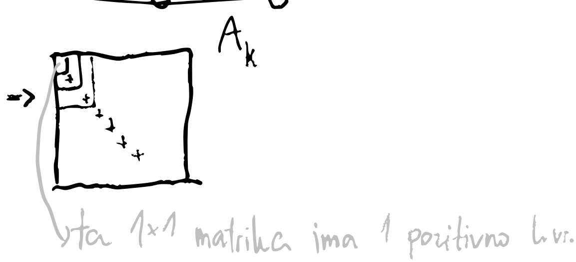
④ A je ND $\Leftrightarrow$ $\lambda_1 \dots \lambda_n < 0$

⑤ A je nedefinitna $\Leftrightarrow$ $\lambda_1 > 0$, $\lambda_2 < 0$



> 0 za $\forall x \in \mathbb{R}^n$

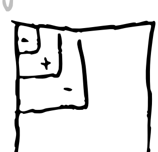
A PD $\Rightarrow$ "levi zgornji vogal" matrike A PD



Izrek (Sylvester)

A PD $\Rightarrow$ $\det(A_k) > 0$ za $k = 1, 2, \dots, n$

velja tudi v $\Leftarrow$

A ND $\Rightarrow$ 

A ND $\Rightarrow$ $\det(A_k) > 0$ za sode k, $1 \leq k \leq n$

$\det(A_k) < 0$ za lihe k, $1 \leq k \leq n$

$A = BB^T$?

① BB^T je vedno:
- simetrična
- PSD

② Naj bo $A \in \mathbb{R}^{n \times n}$ PD.

$A = QDQ^T =$ $\qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} > 0$

$= QEEQ^T =$ $\qquad E = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} \quad E^2 = D$

$= (QE)(QE)^T$ $\qquad E = E^T$!

$B := QE$

$A = BB^T$

Izrek:

A obrnljiva $n \times n$ matrika
$A = BB^T$ za neko $B \in \mathbb{R}^{n \times n}$, $\operatorname{rang} B = n$, natanko tedaj, ko je A PD. (in seveda simetrična)

V posebnem:

Naredimo QR faktorizacijo matrike B^T.

$B^T = QR$ $\qquad$ Q ortogonalna, R zgΔ $\qquad$ $L = R^T$ spΔ

A je PD $\Leftrightarrow A = BB^T = R^T \underline{Q^T Q} R = R^T R = LL^T$

Razcep Choleskega

Izrek: (razcep Choleskega)

A obrnljiva matrika

A ima razcep Choleskega ($A = LL^T$ za obrnljivo sp. Δ. matr.)

natanko tedaj, ko je A PD. (in simetrična)