Matematika 1, vaje, 5. 1. 2021

(c) $h(x,y) = (1+e^y)\cos x - ye^y$

Stacionarne točke h so ničle gradienta $\operatorname{grad} h = \begin{bmatrix} h_x \\ h_y \end{bmatrix}$:

$h_x = -(1+e^y)\sin x = 0 \;..\; \sin x = 0 \;....\; x = k\pi,\ k \in \mathbb{Z}$

vstavimo

$h_y = e^y \cos x - (e^y + ye^y) = e^y(\cos x - 1 - y) = 0$

$e^y(\cos(k\pi) - 1 - y) = 0 \;...\; \cos(k\pi) - 1 - y = 0,$

tj. $y = \cos(k\pi) - 1$

Stac. točke h so $T_k(k\pi, \overbrace{\cos(k\pi)}^{(-1)^k} - 1)$ oz. $T_k(k\pi, (-1)^k - 1)$.

Tip teh stac. točk?

$$Hh = \begin{bmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{bmatrix} = \begin{bmatrix} -(1+e^y)\cos x, & -e^y \sin x \\ -e^y \sin x, & e^y\cdot(\cos x - 2 - y) \end{bmatrix}$$

$$h_{yy} = e^y\cdot(\cos x - 1 - y) + e^y\cdot(-1)$$

Vrednost H_h v T_k:

$$Hh(k\pi, (-1)^k - 1) = \begin{bmatrix} -(1+e^{(-1)^k-1})\cdot(-1)^k & 0 \\ 0 & e^{(-1)^k-1}\cdot\underbrace{(\cancel{(-1)^k} - 2 - (\cancel{(-1)^k} - 1))}_{-1} \end{bmatrix} =$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{oz.} \quad \begin{bmatrix} 1+e^{-2} & 0 \\ 0 & -e^{-2} \end{bmatrix}$$

$\uparrow$ sod k $\qquad\qquad$ $\uparrow$ lih k

Pri sodih k ima Hh neg. last. vrednosti, torej je negativno definitna, zato so $T_k(k\pi, 0)$ za sode k lokalni maksimi.

Pri lihih k ima Hh ima neg. in poz. lastne vrednosti, tj. ni definitna, zato so $T_k(k\pi, -2)$ za lihe k sedlaste točke (niso lok. ekstremi).

(d) $k(x,y,z) = x^3 + y^3 + 3z^2 - 3xyz$

$k_x = 3x^2 - 3yz = 0 \ldots \quad 3x^2 - 3y \cdot \frac{1}{2}xy = 0 \ldots \quad x^2 - \frac{1}{2}xy^2 = 0 \quad (*)$

$k_y = 3y^2 - 3xz = 0 \ldots \quad 3y^2 - 3x \cdot \frac{1}{2}xy = 0 \ldots \quad y^2 - \frac{1}{2}x^2y = 0 \quad (**)$

$k_z = 6z - 3xy = 0 \ldots \quad z = \frac{1}{2}xy$

$x\left(x - \frac{1}{2}y^2\right) = 0$

torej $x = 0$ ali $x - \frac{1}{2}y^2 = 0$

iz $(**)$ dobimo	t.j. $x = \frac{1}{2}y^2$ /z $(**)$:
$y^2 = 0 \ldots y = 0$	$y^2 - \frac{1}{2}\left(\frac{1}{2}y^2\right)^2 y = 0$
~~zato~~ $z = \frac{1}{2}\cdot 0 \cdot 0 = 0$	$y^2\left(1 - \frac{1}{8}y^3\right) = 0$
$\underline{T_1(0,0,0)}$	$y = 0$ ali $1 - \frac{y^3}{8} = 0$

oz. $y = 0$ ali $y = 2$

$x = \frac{1}{2}y^2 = 0$	$x = \frac{1}{2}2^2 = 2$
$z = \frac{1}{2}xy = 0$	$z = \frac{1}{2}\cdot 2 \cdot 2 = 2$
	$\underline{T_2(2,2,2)}$

Tip teh stac. točk:

$$H_k = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} = \begin{bmatrix} 6x & -3z & -3y \\ -3z & 6y & -3x \\ -3y & -3x & 6 \end{bmatrix}$$

$$H_k(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ ← last. vred. so $0, 0, 6$, mat. je le semidefinitna, na podlagi H_k tipa stac. točke ne moremo določit.

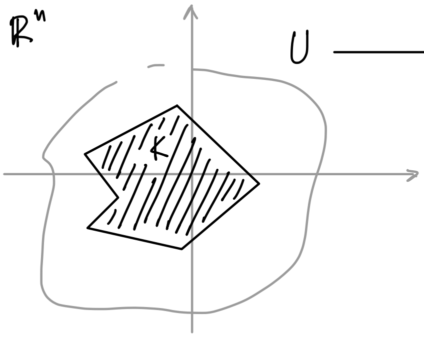
$$H_k(2,2,2) = \begin{bmatrix} 12 & -6 & -6 \\ -6 & 12 & -6 \\ -6 & -6 & 6 \end{bmatrix}$$

$12 > 0$

$\begin{vmatrix} 12 & -6 \\ -6 & 12 \end{vmatrix} = 144 - 36 > 0$

$\det(H_k(2,2,2)) < 0$, mat. ni definitna, stac. točka T_2 je torej $\underline{\text{sedlo}}$.

3. Poišči tiste točke na elipsi z enačbo
$$x^2 - xy + y^2 = 3,$$
ki so najbolj oddaljene od koordinatnega izhodišča.



$U \xrightarrow{f} \mathbb{R}$

Kaj je največja in najmanjša vrednost, ki jo f zavzame na K?

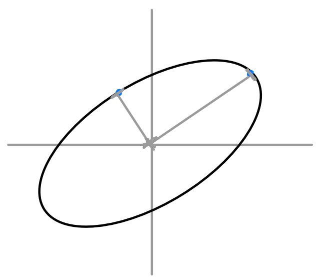
(Če je K $\underbrace{\text{omejena in vsebuje svoj rob}}_{\text{zaprta v } \mathbb{R}^n}$, f pa zvezna, potem f zavzame min. in max. vrednost.)

K bo dana s sistemom (ne)enačb. Če je dana z eno samo enačbo, recimo $\underbrace{g(\vec{x})=0}_{\text{vez.}}$, potem so kandidati za globalne ekstreme stac. točke pripadajoče Lagrangeove funkcije:

$$L(\vec{x},\lambda) = f(\vec{x}) - \lambda g(\vec{x}).$$

V našem primeru: K je opisana z en. $x^2 - xy + y^2 = 3$

... $\underbrace{x^2 - xy + y^2 - 3}_{g(x,y)} = 0$



Funkcija: $f(x,y) = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

Pripadajoča Lagrangeova funkcija:

$$L(x,y,\lambda) = x^2+y^2 - \lambda(x^2-xy+y^2-3)$$

$L_x = 2x - \lambda(2x-y) = 0 \quad \ldots \quad \lambda = \dfrac{2x}{2x-y}$

$L_y = 2y - \lambda(2y-x) = 0 \ldots \quad \lambda = \dfrac{2y}{2y-x}$

$L_\lambda = -(x^2-xy+y^2-3) = 0$

$\dfrac{2x}{2x-y} = \dfrac{2y}{2y-x} \quad / \cdot (2x-y)(2y-x)$

$2x(2y-x) = 2y(2x-y)$

$\cancel{4xy} - 2x^2 = \cancel{4xy} - 2y^2$

$x^2 = y^2$ oz. $y = \pm x$

$y = \pm x$ vstavimo v en. vezi $g(x,y) = 0$ (oz. $L_\lambda = 0$):

$y = x$: $\cancel{x^2} - \cancel{x^2} + x^2 - 3 = 0 \ \dots \ x = \pm\sqrt{3}$, $y = \pm\sqrt{3}$, $T_1(\sqrt{3}, \sqrt{3})$, $T_2(-\sqrt{3}, -\sqrt{3})$

$y = -x$: $x^2 - x\cdot(-x) + x^2 - 3 = 0 \dots 3x^2 = 3 \dots x = \pm 1$, $y = \mp 1$

$T_3(1,-1)$, $T_4(-1,1)$

To so kandidati za globalne ekstreme f pri pogoju $g(x,y) = 0$.

V f sedaj vstavimo te stac. točke:

(x,y)	$f(x,y) = x^2 + y^2$
T_1	6
T_2	6
T_3	2
T_4	2

$\leftarrow$ T_1 in T_2 sta najbolj oddaljeni (na razdalji $\sqrt{6}$),

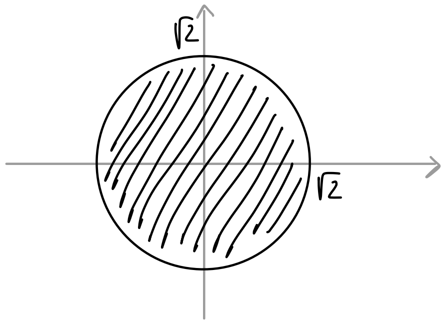
$\leftarrow$ T_3 in T_4 sta najbližji $(0,0)$ (na razdalji $\sqrt{2}$).

5. Poišči največjo in najmanjšo vrednost funkcije $f(x,y) = xy - x + y - 1$

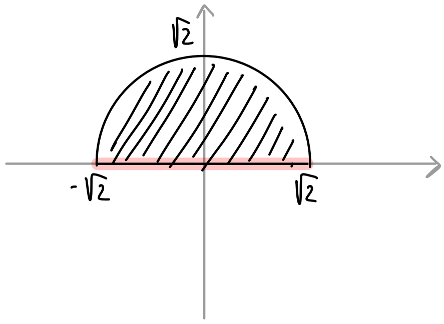
(a) na krogu danem z $x^2 + y^2 \le 2$,

(b) na polkrogu danem z $x^2 + y^2 \le 2$ in $y \ge 0$.

(a)



(b)



(a) Ločimo rob in notranjost kroga (območja):

- Rob je dan z en. $x^2+y^2=2 \ \dots \ x^2+y^2-2=0$

Pripadajoča L. funk. je $L(x,y,\lambda) = f(x,y) - \lambda(x^2+y^2-2) =$

$$= xy - x + y - 1 - \lambda(x^2+y^2-2)$$

$$L_x = y - 1 - \lambda\, 2x = 0 \ \dots \ \lambda = \frac{y-1}{2x}$$

$$L_y = x + 1 - \lambda \cdot 2y = 0 \ \dots \ \lambda = \frac{x+1}{2y}$$

$$L_\lambda = -(x^2+y^2-2) = 0 \quad (*)$$

$$\frac{y-1}{2x} = \frac{x+1}{2y} \ \dots \ (y-1)y = (x+1)x$$

$$y^2 - y = x^2 + x$$

$$y^2 - x^2 - y - x = 0$$

$$(y-x)(y+x) - (y+x) = 0$$

$$(y-x-1)(y+x) = 0$$

Torej: $y - x - 1 = 0$

$y = x + 1$, vstavimo v $(*)$:

$$x^2 + (x+1)^2 - 2 = 0$$

$$2x^2 + 2x - 1 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{12}}{4} =$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

$$y_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

$T_1\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}, \frac{1}{2}+\frac{\sqrt{3}}{2}\right)$, $T_2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}, \frac{1}{2}-\frac{\sqrt{3}}{2}\right)$

ali

$y + x = 0$

$y = -x$, vstavimo v $(*)$:

$$x^2 + x^2 - 2 = 0 \ \dots \ 2x^2 = 2$$

$$x = \pm 1$$

$$y = \mp 1$$

$T_3(1,-1)$

$T_4(-1,1)$

• notranjost: $x^2+y^2<2$; poiščemo stac. točke f, ki ustrezajo tej neenačbi:

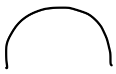
$f_x = y-1 = 0 \; \dots \; y = 1$

$f_y = x+1 = 0 \; \dots \; x = -1$

$T_5(-1,1)$, ali ta leži v notranjosti?
$(-1)^2 + 1^2 = 2 \not< 2$, kandidatov za ekstreme v notranjosti ni.

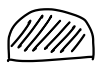
Vrednosti f v kandidatih za globalne ekstreme:

(x,y)	$f(x,y) = xy - x + y - 1$	
T_1	$1/2$	← največja vrednost
T_2	$1/2$	
T_3	-4	← najmanjša vrednost
T_4	0	

(b) Poleg ekstremov na  delu roba (označeni z na prejšnji strani), preverimo kaj so kandidati za ekstreme na ______ delu roba.

To je interval $[-\sqrt{2}, \sqrt{2}]$ na x-osi (tj. $y=0$)

$f(x,y) = xy - x + y - 1 \; \dots \; f(x,0) = -x-1$, min./max. vrednost te funk. 1 spremenljivke na $[-\sqrt{2}, \sqrt{2}]$ pa je $\underline{-\sqrt{2}-1}$ / $\underline{\sqrt{2}-1}$

Torej je $-\sqrt{2}-1$ najmanjša, $\frac{1}{2}$ pa največja vrednost f na .