Matematika 1, vaje, 24.11.2020

1. Preslikava $\tau\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ je podana s predpisom

$$\tau(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X + X \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad , \quad K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   (a) Pokaži, da je τ linearna preslikava.

   (b) Določi njeno matriko v bazi $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ prostora $\mathbb{R}^{2\times 2}$.

(a) $\tau(X) = KX + XK$

$$\tau(\alpha X + \beta Y) = K(\alpha X + \beta Y) + (\alpha X + \beta Y)K =$$

$$= \alpha KX + \beta KY + \alpha XK + \beta YK = \alpha \underbrace{(KX + XK)}_{\overset{\shortparallel}{\tau(X)}} + \beta \underbrace{(KY + YK)}_{\overset{\shortparallel}{\tau(Y)}},$$

tj. τ je linearna.

(b)

$$\tau(E_{11}) = KE_{11} + E_{11}K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\tau(E_{12}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\tau(E_{21}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\tau(E_{22}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0\cdot E_{11} + 1\cdot E_{12} + 1\cdot E_{21} + 0\cdot E_{22}$$

$$A_\tau = \begin{array}{cccc|c} E_{11} & E_{12} & E_{21} & E_{22} & \\ 2 & 1 & 1 & 0 & E_{11} \\ 1 & 1 & 0 & 1 & E_{12} \\ 1 & 0 & 1 & 1 & E_{21} \\ 0 & 1 & 1 & 0 & E_{22} \end{array}$$

$$\tau(I) = KI + IK = 2K = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$I = E_{11} + E_{22} \longleftrightarrow \vec{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

glede na std. bazo $\mathbb{R}^{2\times 2}$

$$A_\tau \vec{e} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = \tau(I)$$

5. (a) Ali je množica $\{p(x) = ax + b : a \neq 0, a, b \in \mathbb{R}\}$ vektorski podprostor v vektorskem prostoru polinomov $\mathbb{R}_1[x]$?

(b) Ali je množica $\{p(x) : p(1) = 0\}$ vektorski podprostor v vektorskem prostoru polinomov $\mathbb{R}_2[x]$?

(c) Ali je množica $\{p(x) : p(0) = 1\}$ vektorski podprostor v vektorskem prostoru polinomov $\mathbb{R}_n[x]$?

(d) Ali je množica $\{p(x) : p''(3) = 0\}$ vektorski podprostor v vektorskem prostoru polinomov $\mathbb{R}_3[x]$?

Poišči še baze podprostorov.

(a) U_a ← polinomi st. točno 1 v $\mathbb{R}_1[x]$

Ali U_a vsebuje ničelni polinom $0\cdot x+0$? Ne, torej U_a ni vekt. podprostor.

(b) $U_b = \{p(x) : p(1) = 0\} \subseteq \mathbb{R}_2[x]$

Preverimo, da je vekt. podprostor:

$p, q \in U_b$ .... $p(1) = 0$ in $q(1) = 0$

$(\alpha p + \beta q)(1) = \alpha p(1) + \beta q(1) = \alpha \cdot 0 + \beta \cdot 0 = 0$, tj. $\alpha p + \beta q \in U_b$,

U_b je vekt. podprostor v $\mathbb{R}_2[x]$.

Kaj je baza U_b?

V U_b so vsi polinomi (st. max. 2), ki imajo v $x = 1$ ničlo;

npr. $p_1(x) = x - 1$ in $p_2(x) = x(x-1)$ (ali $p_2(x) = x^2 - 1$)

Polinomov višjih stopenj v U_b ni.

Kako "izgledajo" vsi polinomi iz U_b? $p(x) = (ax+b)(x-1)$

(ali $p(x) = ax^2 + bx + c$ ... $p(1) = 0$, torej $a + b + c = 0$
oziroma $c = -a - b$, zato $p(x) = ax^2 + bx - a - b$.)

Bazo lahko izberemo tako, da proste parametre postavimo na 0, razen enega, ki ga postavimo na 1.

Torej $B_{U_b} = \{x(x-1), x-1\}$.

(ali $B_{U_b} = \{x^2 - 1, x - 1\}$)

(c) $U_c = \{p \in \mathbb{R}_n[x] : p(0) = 1\}$, ta spet ne vsebuje ničelnega polinoma (če $p(x) = 0$, potem tudi $p(0) = 0$), torej U_c ni vekt. podprostor.

(d) $U_d = \{p \in \mathbb{R}_3[x] : p''(3) = 0\}$

$p, q \in U_d \ldots \quad p''(3) = 0$ in $q''(3) = 0$

$(\alpha p + \beta q)''(3) = \alpha p''(3) + \beta q''(3) = \alpha \cdot 0 + \beta \cdot 0 = 0$, tj.

$\alpha p + \beta q \in U_d$, torej je U_d vektorski podprostor v $\mathbb{R}_3[x]$.

$1 \in U_d$, tj. $p(x) = 1$. (Saj $p''(x) = 0$, zato $p''(3) = 0 \ldots$)
$x \in U_d$, tj. $p(x) = x$. (______ " ______ ..)

Iščemo še en polinom $\in U_d$, ki je lin. neodvisen od 1 in x ....

Recimo $p''(x) = x - 3 \ldots \quad p'(x) = \int (x-3)\,dx = \frac{x^2}{2} - 3x$

$$p(x) = \int\left(\frac{x^2}{2} - 3x\right)dx = \frac{x^3}{6} - \frac{3x^2}{2}$$

Preverimo, da je $\left\{1, x, \frac{x^3}{6} - \frac{3x^2}{2}\right\}$ res baza ~~za~~ U_d.

7. *(Prostor polinomov)* Naj bo $\mathbb{R}[x]$ množica vseh polinomov z realnimi koeficienti v spremenljivki x. ($\mathbb{R}[x]$ torej vsebuje polinome vseh stopenj!) Preveri, da je $\mathbb{R}[x]$ vektorski prostor za običajni operaciji seštevanja polinomov in množenja s skalarjem. Poišči bazo za $\mathbb{R}[x]$. Poišči še bazo podprostora

$$W = \{p \in \mathbb{R}[x] : p(1) = p(-1) = 0\}.$$

Koliko je $\dim \mathbb{R}[x]$ in koliko $\dim W$?

Kaj je baza $\mathbb{R}[x]$? $\quad B_{\mathbb{R}[x]} = \{1, x, x^2, x^3, x^4, x^5, x^6, \ldots\}$

Zakaj je W res podprostor v $\mathbb{R}[x]$?

$p, q \in W \ldots \quad p(1) = p(-1) = 0$ in $q(1) = q(-1) = 0$

$(\alpha p + \beta q)(\pm 1) = \alpha p(\pm 1) + \beta q(\pm 1) = \alpha \cdot 0 + \beta \cdot 0 = 0$, tj. $\alpha p + \beta q \in W$,

W je torej vektorski podprostor.

$x^2 - 1$ ima 1 in -1 za ničli, torej $x^2 - 1 \in W$
$x(x^2-1) \in W$
$x^2(x^2-1) \in W$
$\vdots$

$B_W = \{x^2 - 1, x(x^2-1), x^2(x^2-1), \ldots\}$, $\dim W = \infty$.

3. V $\mathbb{R}^3$ so dani vektorji $\mathbf{a} = [1,1,0]^\mathsf{T}$, $\mathbf{b} = [1,0,1]^\mathsf{T}$ in $\mathbf{c} = [0,1,1]^\mathsf{T}$ ter linearna preslikava $\tau : \mathbb{R}^3 \to \mathbb{R}^3$, za katero velja

$$\tau(\mathbf{a}) = \mathbf{a}, \tau(\mathbf{b}) = \mathbf{a} + \mathbf{b} \text{ ter } \tau(\mathbf{c}) = \mathbf{a} + \mathbf{c}.$$

(a) Pokaži, da je $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ baza prostora $\mathbb{R}^3$.

(b) Zapiši matriko preslikave τ v bazi $\mathcal{B} := \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

(c) Zapiši matriko preslikave τ v standardni bazi $\mathcal{S} := \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

(d) Kam preslika τ vektor $[1,1,1]^\mathsf{T}$?

(a) Ker so $\vec{a}$, $\vec{b}$ in $\vec{c}$ trije, bodo tvorili bazo $\mathbb{R}^3$ ($\dim \mathbb{R}^3 = 3$), če so linearno neodvisni.

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{Gauss}} \underset{\text{elim.}}{\cdots\cdots} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ ki je polnega ranga, torej so $\vec{a}, \vec{b}, \vec{c}$ lin. neodv., tj. $\{\vec{a}, \vec{b}, \vec{c}\}$ baza $\mathbb{R}^3$.

(b)
$$A_{\tau,\mathcal{B},\mathcal{B}} = \begin{array}{c|c} \begin{matrix} \vec{a} & \vec{b} & \vec{c} \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{matrix} \end{array}.$$

(c) $A_{\tau,\mathcal{S},\mathcal{S}} = ?$, $\mathcal{S} = \{\vec{i}, \vec{j}, \vec{k}\}$

$$\tau(\vec{i}) = \tau\left(\tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{b} - \tfrac{1}{2}\vec{c}\right) = \tfrac{1}{2}\tau(\vec{a}) + \tfrac{1}{2}\tau(\vec{b}) - \tfrac{1}{2}\tau(\vec{c}) = \frac{\vec{a}}{2} + \frac{\vec{a}+\vec{b}}{2} - \frac{\vec{a}+\vec{c}}{2} = \frac{\vec{a}+\vec{b}-\vec{c}}{2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\tau(\vec{j}) = \tfrac{1}{2}\tau(\vec{a}) - \tfrac{1}{2}\tau(\vec{b}) + \tfrac{1}{2}\tau(\vec{c}) = \frac{\vec{a}}{2} - \frac{\vec{a}+\vec{b}}{2} + \frac{\vec{a}+\vec{c}}{2} = \frac{\vec{a}-\vec{b}+\vec{c}}{2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\tau(\vec{k}) = -\tfrac{1}{2}\tau(\vec{a}) + \tfrac{1}{2}\tau(\vec{b}) + \tfrac{1}{2}\tau(\vec{c}) = -\frac{\vec{a}}{2} + \frac{\vec{a}+\vec{b}}{2} + \frac{\vec{a}+\vec{c}}{2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Ker je $\mathcal{B} = \{\vec{a}, \vec{b}, \vec{c}\}$ baza $\mathbb{R}^3$, lahko $\vec{i}$, $\vec{j}$ in $\vec{k}$ zapišemo v tej bazi:

$$\vec{i} = \alpha_1\vec{a} + \beta_1\vec{b} + \gamma_1\vec{c} = [\vec{a}, \vec{b}, \vec{c}]\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}$$

$$\vec{k} = [\vec{a}, \vec{b}, \vec{c}]\begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}$$

$$\vec{j} = \alpha_2\vec{a} + \beta_2\vec{b} + \gamma_2\vec{c} = [\vec{a}, \vec{b}, \vec{c}]\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}$$

$$\left[\vec{a}, \vec{b}, \vec{c}\right]\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \left[\vec{i}, \vec{j}, \vec{k}\right] = I, \quad \text{tj.} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \left[\vec{a}, \vec{b}, \vec{c}\right]^{-1}.$$

$$\begin{array}{ccc|ccc} \vec a & \vec b & \vec c & \vec i & \vec j & \vec k \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & +1 & 0 & +1/2 & -1/2 & +1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

vstavimo

$$A_{\tau,\mathcal{S},\mathcal{S}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) $$\tau\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = A_{\tau,\mathcal{S},\mathcal{S}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Kako izrazimo $\tau\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, če ne poznamo $A_{\tau,\mathcal{S},\mathcal{S}}$?

Izrazimo $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ v bazi $B = \{\vec a, \vec b, \vec c\}$:

$\vec a + \vec b + \vec c = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, torej je $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\vec a + \frac{1}{2}\vec b + \frac{1}{2}\vec c$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}_B \quad \dots \quad A_{\tau,B,B} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}_B = 1\cdot\vec a + 1\cdot\vec b + \frac{1}{2}\vec c = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{S}}.$$