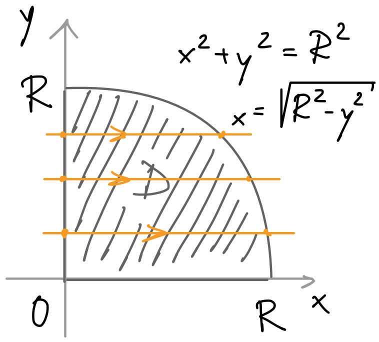
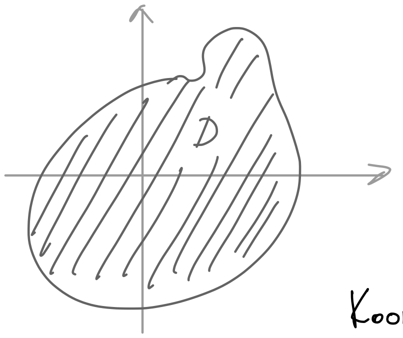
Matematika 1, vaje, 22. 12. 2020



1. Poišči koordinate masnega središča četrtine kroga; $x^2+y^2 \leq R^2$, $x \geq 0$, $y \geq 0$, če je gostota v vsaki točki enaka oddaljenosti od izhodišča, tj. $\rho(x,y)=\sqrt{x^2+y^2}$.
Namig: masa lika $D \subseteq \mathbb{R}^2$ je dana z $m=\iint_D \rho(x,y)dxdy$, koordinati masnega središča pa sta $x^* = \frac{1}{m}\iint_D x\rho(x,y)dxdy$ in $y^* = \frac{1}{m}\iint_D y\rho(x,y)dxdy$. Uvedi polarne koordinate.



$$m = \iint_D \rho(x,y)\,dx\,dy$$

↑ to je masa območja D s površinsko gostoto $\rho(x,y)$

Koordinati masnega središča sta:

$$x^* = \frac{1}{m}\iint_D x\,\rho(x,y)\,dx\,dy\,, \quad y^* = \frac{1}{m}\iint_D y\,\rho(x,y)\,dx\,dy\,.$$

Poskusimo (naivno) v kartezičnih koordinatah:

$$m = \int_0^R \left(\int_0^{\sqrt{R^2-y^2}} \sqrt{x^2+y^2}\,dx \right) dy \;\ldots\ldots$$

Kaj pa, če uvemo polarne koordinate? $\quad x = r\cos\varphi,\ y = r\sin\varphi \quad \det(JF) = r$

$$m = \iint_D \rho(x,y)\,dx\,dy = \int_0^{\pi/2}\left(\int_0^R \underbrace{\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}}_{r} \cdot r\,dr\right)d\varphi =$$

$$= \int_0^{\pi/2}\left(\int_0^R r^2\,dr\right)d\varphi = \int_0^{\pi/2} \underbrace{\left(\frac{r^3}{3}\Big|_{r=0}^{r=R}\right)}_{\frac{R^3}{3}} d\varphi = \frac{R^3}{3}\cdot\frac{\pi}{2} = \frac{\pi R^3}{6}.$$

$$x^* = \frac{1}{m}\iint_D x\,\rho(x,y)\,dx\,dy = \frac{6}{\pi R^3}\int_0^{\pi/2}\left(\int_0^R r\cos\varphi \cdot r^2\,dr\right)d\varphi =$$

$$= \frac{6}{\pi R^3}\int_0^{\pi/2} \cos\varphi \underbrace{\left(\frac{r^4}{4}\right)\Big|_{r=0}^{r=R}}_{\frac{R^4}{4}} d\varphi = \frac{\cancel{6}\,3}{\pi \cancel{R^3}}\cdot\frac{\cancel{R^4}}{\cancel{4}\,2}\cdot \underbrace{\sin\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}}}_{1} = \frac{3R}{2\pi}$$

$$y^* = \frac{6}{\pi R^3}\int_0^{\pi/2}\left(\int_0^R \overbrace{r\sin\varphi}^{y}\cdot r^2\,dr\right)d\varphi = \frac{6}{\pi R^3}\overbrace{\left(\int_0^R r^3\,dr\right)}^{R^4/4}\overbrace{\left(\int_0^{\pi/2}\sin\varphi\,d\varphi\right)}^{1} = \frac{3R}{2\pi}$$

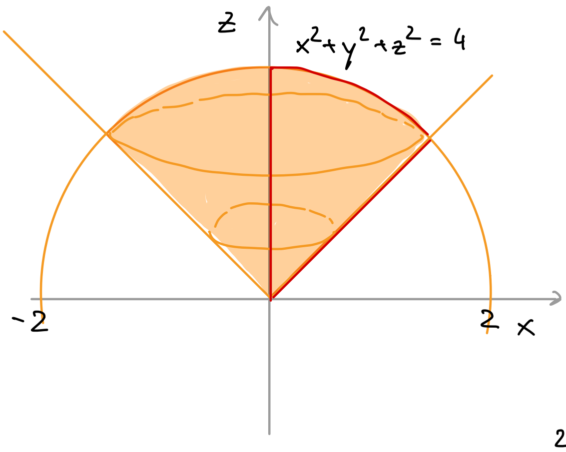
$$\int_{\varphi_1}^{\varphi_2}\left(\int_{R_1}^{R_2} f(r)\cdot g(\varphi)\,dr\right)d\varphi = \left(\int_{R_1}^{R_2} f(r)\,dr\right)\cdot\left(\int_{\varphi_1}^{\varphi_2} g(\varphi)\,d\varphi\right)$$

Koordinati masnega središča sta $(x^*, y^*) = \left(\frac{3R}{2\pi}, \frac{3R}{2\pi}\right)$.

2. Določi maso in koordinate masnega središča homogenega telesa (tj. $\rho(x,y,z) = 1$), ki je omejeno s ploskvama $z^2 = x^2 + y^2$ ter $x^2 + y^2 + z^2 = 4$ in leži v polprostoru $z \geq 0$.
Namig: Vpelji ti. sferne oz. krogelne koordinate:

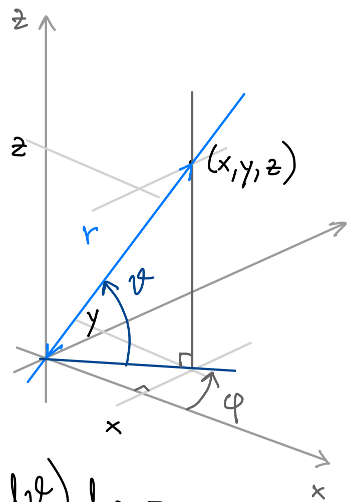
$$x = r\cos\theta\cos\varphi,$$
$$y = r\cos\theta\sin\varphi,$$
$$z = r\sin\theta,$$

tj. 'novo spremenljivko' $\mathbf{F}(r,\varphi,\theta) = [x,y,z]^{\mathsf{T}}$ (za katero je $\det(J\mathbf{F}) = r^2\cos\theta$.)



$$x = r\cos\vartheta\cos\varphi$$
$$y = r\cos\vartheta\sin\varphi$$
$$z = r\sin\vartheta$$

$$\det J\vec{F} = r^2\cos\vartheta$$



$$\underline{m} = \iiint_D \underbrace{\rho(x,y,z)}_{1}\,dx\,dy\,dz = \int_0^{2\pi}\left(\int_{\pi/4}^{\pi/2}\left(\int_0^2 1\cdot r^2\cos\vartheta\,dr\right)d\vartheta\right)d\varphi =$$

$$= \underbrace{\left(\int_0^{2\pi} d\varphi\right)}_{2\pi}\underbrace{\left(\int_{\pi/4}^{\pi/2}\cos\vartheta\,d\vartheta\right)}_{\sin\vartheta\Big|_{\vartheta=\pi/4}^{\vartheta=\pi/2}}\cdot\underbrace{\left(\int_0^2 r^2\,dr\right)}_{\frac{r^3}{3}\Big|_{r=0}^{r=2}} = 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{8}{3} = \underline{\frac{8\pi}{3}(2-\sqrt{2})}$$

$$z^* = \frac{1}{m}\iiint_D z\cdot\rho(x,y,z)\,dx\,dy\,dz = \frac{1}{m}\int_0^{2\pi}\left(\int_{\pi/4}^{\pi/2}\left(\int_0^2 \underbrace{r\sin\vartheta\cdot 1\cdot r^2\cos\vartheta}_{r^3\sin\vartheta\cdot\cos\vartheta}\,dr\right)d\vartheta\right)d\varphi =$$

$$= \frac{1}{m}\underbrace{\left(\int_0^{2\pi} d\varphi\right)}_{2\pi}\cdot \underbrace{\left(\int_{\frac{\pi}{4}}^{\pi/2} \sin\vartheta\cos\vartheta\, d\vartheta\right)}_{\|}\cdot \underbrace{\left(\int_0^2 r^3\, dr\right)}_{4} = \frac{2\pi}{m} = \underbrace{\frac{3}{4(2-\sqrt{2})}}_{\substack{\| \\ z^*}}.$$

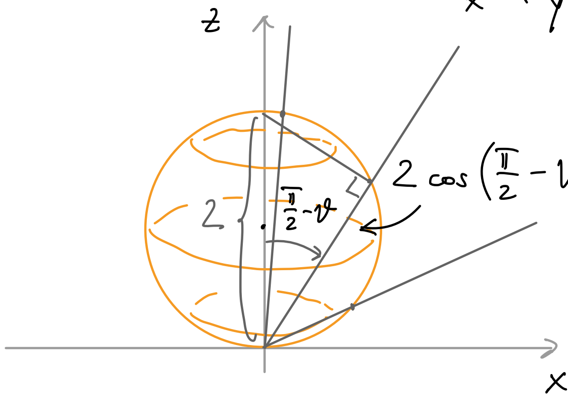
$t = \sin\vartheta$
$dt = \cos\vartheta\, d\vartheta$

$$\int_{\sqrt{2}/2}^{1} t\, dt = \frac{t^2}{2}\Big|_{t=\sqrt{2}/2}^{t=1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$x^* = \frac{1}{m}\underbrace{\left(\int_0^{2\pi} \cos\varphi\, d\varphi\right)}_{0}\cdot \;\ldots\; = \underline{0} \quad\ldots\quad y^* = \underline{0}.$$

4. Določi maso in koordinate masnega središča krogle z neenačbo $x^2 + y^2 + z^2 \leq 2z$, če je njena gostota v vsaki točki enaka oddaljenosti od izhodišča.
Namig: Uvedi krogelne koordinate.

$$x^2 + y^2 + z^2 \leq 2z \quad\ldots\quad x^2 + y^2 + z^2 - 2z \leq 0 \quad / + 1$$
$$x^2 + y^2 + \underbrace{z^2 - 2z + 1}_{(z-1)^2} \leq 1 \leftarrow$$ to je krogla s polmerom 1 in središčem v $(0,0,1)$.



$$2\cos\left(\frac{\pi}{2} - \vartheta\right) = 2\sin\vartheta$$

$$m = \int_0^{2\pi}\left(\int_0^{\pi/2}\left(\int_0^{2\sin\vartheta} r\cdot r^2\cos\vartheta\, dr\right) d\vartheta\right) d\varphi =$$

$$\rho(x,y,z) = \sqrt{x^2+y^2+z^2} = r$$

$$= \int_0^{2\pi}\left(\int_0^{\pi/2} \cos\vartheta \cdot \frac{r^4}{4}\Big|_{r=0}^{r=2\sin\vartheta} d\vartheta\right) d\varphi =$$

$t = \sin\vartheta,\ dt = \cos\vartheta\, d\vartheta$

$$= 4\int_0^{2\pi}\left(\int_0^{\pi/2} \cos\vartheta\cdot\sin^4\vartheta\, d\vartheta\right) d\varphi =$$

$$= 4\cdot 2\pi \int_0^1 t^4\, dt = \underline{\frac{8}{5}\pi}.$$

$z^* = \frac{1}{m}\cdot \ldots$ samostojno doma!

6. Poišči in kasificiraj stacionarne točke spodnjih funkcij.

(a) $f(x,y) = x^3 - 4x^2 + 2xy - y^2$
(b) $g(x,y) = xe^x + 2ye^y + 1$
(c) $h(x,y) = (1+e^y)\cos x - ye^y$

(a) $f(x,y) = x^3 - 4x^2 + 2xy - y^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 8x + 2y = 0 \quad \ldots \quad 3x^2 - 8x + 2x = 0 \ldots \; 3x(x-2) = 0$$

$\uparrow$ vstavimo

$x_1 = 0, \; x_2 = 2$

$y_1 = 0, \; y_2 = 2$

$$\frac{\partial f}{\partial y} = 2x - 2y = 0 \quad \ldots \quad x = y$$

f ima 2 stacionarni točki $T_1(0,0)$, $T_2(2,2)$.

Ali sta ti dve točki tudi lok. ekstrema?

Tip stac. točke dobimo z uporabo Hessejeve mat. f:

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x-8, & 2 \\ 2, & -2 \end{bmatrix}$$

V stac. točkah je:

T_1 : $Hf(0,0) = \begin{bmatrix} -8 & 2 \\ 2 & -2 \end{bmatrix}$ $\left.\begin{array}{l} -8 < 0 \\ \det Hf(0,0) = 12 > 0 \end{array}\right\}$ je neg. definitna

Torej je T_1 lokalni maksimum.

T_2 : $Hf(2,2) = \begin{bmatrix} 4 & 2 \\ 2 & -2 \end{bmatrix}$ $\left.\begin{array}{l} 4 > 0 \\ \det(Hf(2,2)) = -12 \end{array}\right\}$ ni definitna

Torej T_2 ni lokalni ekstrem (je sedlasta točka).