Matematika 1, vaje, 20.10.2020

Schurov razcep matrike $A \in \mathbb{R}^{n\times n}$ je zapis $A = QZQ^T$.

ortogonalna, $Q^TQ = I$ — zgornje trikotna

2. Poišči Schurova razcepa matrik

$$A = \begin{bmatrix} 6 & -1 & 1 \\ 4 & 3 & 1 \\ 2 & 2 & 3 \end{bmatrix} \text{ in } B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Začnemo z B. Poiščimo vsaj eno lastno vrednost in pripadajoč lastni vektor B.

$$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ -\sqrt{2} & -\sqrt{2} & \boxed{2-\lambda} \end{vmatrix} = \begin{vmatrix} 2-\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix}(2-\lambda) = (2-\lambda)^2(1-\lambda) = 0.$$

$$\lambda_1 = 1,\ \lambda_{2,3} = 2$$

Poiščimo lastni vektor, ki pripada $\lambda_{2,3} = 2$:

$$B - 2I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ -\sqrt{2} & -\sqrt{2} & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_1$$

$x = 0$
$y = 0$
$z \in \mathbb{R}$

je že normiran

$$Q^T \cdot / B = QZQ^T / \cdot Q \ldots \quad Z = Q^TBQ$$

Prvi stolpec Q je $\underset{\text{normiran}}{\underline{\quad}}$ lastni vektor, ki pripada "prvi" lastni vrednosti.

Vzemimo (zaenkrat): $Q_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\uparrow \vec{v}_1$ | $\uparrow\ \uparrow$ dolžine 1 in pravokotna na $\vec{v}_1$ ter med sabo

$$Z_1 = Q_1^TBQ_1 = \begin{bmatrix} \lambda & \vec{b}^T \\ \vec{0} & B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -\sqrt{2} & 2 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -\sqrt{2} & -\sqrt{2} \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

B_1

Vse skupaj ponovimo na bloku $\square$.

$B_1 = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$, lastni vrednosti B_1 sta 1 in 2.

$\vec{v}_1' = \begin{bmatrix}1\\1\end{bmatrix}$ je lastni vektor B_1 za l. v. 1. $\left(\vec{v}_1 = \begin{bmatrix}0\\1\end{bmatrix} \text{ za l. v. } 2\right)$

Normiramo, dobimo $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$.

$$Q_2' = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & +1\end{bmatrix} \quad \ldots\ldots \quad Z_2' = Q_2'^T B_1 Q_2' = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} =$$

$\vec{v}_1$ pravokoten na $\vec{v}_1$ in dolžine 1

$$= \frac{1}{2}\begin{bmatrix}0 & 2\\ -2 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 2\\ 0 & 4\end{bmatrix} = \begin{bmatrix}1 & 1\\ \boxed{0} & 2\end{bmatrix}$$

Kako to 'zložimo skupaj'?

$$Q_2 := \begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q_2'\end{bmatrix}, \text{ tedaj } Q_2^T \overbrace{\begin{bmatrix}\lambda & \vec{b}^T\\ \vec{0} & B_1\end{bmatrix}}^{Z_1} Q_2 = \begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q_2'^T\end{bmatrix}\begin{bmatrix}\lambda & \vec{b}^T\\ \vec{0} & B_1\end{bmatrix}\begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q_2'\end{bmatrix} =$$

$$= \begin{bmatrix}\lambda & \vec{b}^T\\ \vec{0} & Q_2'^T B_1\end{bmatrix}\begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q_2'\end{bmatrix} = \begin{bmatrix}\lambda & \vec{b}^T Q_2'\\ \vec{0} & Q_2'^T B_1 Q_2'\end{bmatrix} = \begin{bmatrix}\lambda & \vec{b}^T Q_2'\\ \vec{0} & Z_2'\end{bmatrix} = Z_2 = Z \leftarrow$$ iz Schurovega razcepa

(V našem primeru bo to že konec rekurzije.)

Konkretno:

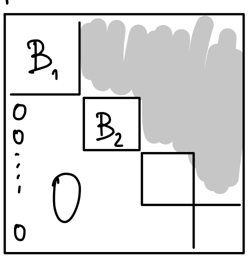
$$Z = Z_2 = \begin{bmatrix}2 & -2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad \vec{b}^T Q_2' = [-\sqrt{2}, -\sqrt{2}]\,\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = [-2, 0]$$

Kaj pa Q iz Schurovega razcepa?

$$Z = Z_2 = Q_2^T Z_1 Q_2 = Q_2^T Q_1^T B \overbrace{Q_1 Q_2}^{Q}, \text{ tj.}$$

$$Q = Q_1 \begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q_2'\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 1\\ 0 & 1 & -1\\ \sqrt{2} & 0 & 0\end{bmatrix}.$$

Kaj je Schurov razcep, če je B bločno zg. trikotna?



B_1 ima l. vred. λ_1 z l. vekt. $\vec{v}_1' = \|$

$$\vec{v}_1 = \begin{bmatrix}\vec{v}_1\\ 0\\ \vdots\\ 0\end{bmatrix}$$

4. Naj bo A poljubna matrika, U in V pa taki ortogonalni matriki, da obstaja produkt UAV. Preveri, da velja naslednje:

(a) $\|UA\|_F = \|A\|_F$,

(b) $\|AV\|_F = \|A\|_F$,

(c) $\|UAV\|_F = \|A\|_F$.

Frobeniusova norma matrike $A \in \mathbb{R}^{m\times n}$ je

$$\|A\|_F = \sqrt{\operatorname{tr}(A^T A)} = \sqrt{\text{"vsota kvadratov elementov } A\text{"}}.$$

(a) Če je U ortogonalna (tj. $U^T U = I$), potem $\|UA\|_F = \|A\|_F$.

(Že vemo $\|U\vec{x}\| = \|\vec{x}\|$, če je U ortogonalna.)

$$\|UA\|_F^2 = \operatorname{tr}\left((UA)^T(UA)\right) = \operatorname{tr}(A^T \underbrace{U^T U}_{I} A) = \operatorname{tr}(A^T A) = \|A\|_F^2.$$

(b) $$\|AV\|_F = \|(AV)^T\|_F = \|\underset{\uparrow \\ \text{ortogonalna}}{V^T} A^T\|_F \underset{\uparrow \\ (a)}{=} \|A^T\|_F = \|A\|_F$$

(c) $$\|U(AV)\|_F \underset{\uparrow \\ (a)}{=} \|AV\|_F \underset{\uparrow \\ (b)}{=} \|A\|_F.$$

5. Poišči matrike ranga 1, ki so (v Frobeniusovi normi) najbližje matrikam:

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Ali so take matrike enolične?

Eckart-Young-ov izrek: Če je $A = USV^T$ SVD matrike A,

$S = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$, U, V ortogonalni, potem je matrika ranga k,

ki je v Frobeniusovi normi najbližja A ravno:

$$A' = U \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_k & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} V^T. \quad (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots)$$

(a) $$\overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A} = I \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} I^T = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} 2 & 0 & 0 \\ 0 & +3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{I^T}_{V^T}$$

Ker iščemo aproksimacijo ranga 1, dobimo:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} I^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ← to je matrika ranga 1, ki je $\|\cdot\|_F$ najbližja A.

(b) $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, B je simetrična, tj. $B = Q \underset{\substack{\uparrow \\ \text{diagonalna}}}{D} Q^T$

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 3^2 = (1-\lambda-3)(1-\lambda+3) =$$

$$= (-\lambda - 2)(4-\lambda) = 0 \quad .. \quad \lambda_1 = 4, \ \lambda_2 = -2.$$ ← lastni vred. B

Lastni vekt.: $\bullet\ \lambda_1 = 4 \ \dots\ B - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \ \dots\ \vec{v}_1 = \begin{bmatrix} y \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$y = \frac{1}{\sqrt{2}}$, da bo ta normiran.

$\bullet\ \lambda_2 = -2 \ \dots\ B + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \ \dots\ \vec{v}_2 = \begin{bmatrix} -y \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

$y = \frac{1}{\sqrt{2}}$

$Q = [\vec{v}_1, \vec{v}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ← ta je ortogonalna

$$B = \underset{\substack{\| \\ U'}}{Q} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \underset{\substack{\| \\ V^T}}{Q^T} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\substack{\| \\ U}} \underset{\substack{\| \\ S}}{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}} \underset{\substack{\| \\ V^T}}{Q^T}.$$

Približek ranga 1 za B je torej:

$$B_1 = U \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} V^T = \underset{\uparrow}{\vec{u}_1} \cdot 4 \cdot \underset{\uparrow}{\vec{v}_1^T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 4 \cdot \frac{1}{\sqrt{2}} [1, 1] = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

prva stolpca U in V

1. Eksponentna funkcija kvadratne $n \times n$ matrike A je (lahko) definirana z

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(V Taylorjevo vrsto za e^x smo namesto števila x vstavili matriko A.)

(a) Utemelji, da velja $\det(e^A) = e^{\operatorname{tr}(A)}$.

(b) Recimo, da je matrika A antisimetrična, tj. $A^\top = -A$. Dokaži, da je tedaj matrika e^A ortogonalna z determinanto 1.

(a) Naj bo $A = QZQ^T$ Schurov razcep matrike A. Tedaj je

$$e^A = e^{QZQ^T} = \sum_{k=0}^{\infty} \frac{1}{k!} (QZQ^T)^k = \sum_{k=0}^{\infty} \frac{1}{k!} Q Z^k Q^T = Q \left(\sum_{k=0}^{\infty} \frac{1}{k!} Z^k \right) Q^T = Q e^Z Q^T.$$

$$(QZQ^T)^k = \underbrace{(QZ\overbrace{\cancel{Q^T})(\cancel{Q}}^{I}Z\cancel{Q^T}) \cdot \ldots \cdot (\cancel{Q}ZQ^T)}_{k\text{-krat}}$$

Poleg tega: Z ima na diagonali lastne vrednosti λ_i matrike A in ker je Z zgornje trikotna, velja, da ima Z^k na diagonali λ_i^k:

$$Z = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad Z^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix}.$$

Torej:

$$e^Z = I + Z + \frac{Z^2}{2} + \ldots + \frac{Z^k}{k!} + \ldots =$$

$$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} + \ldots + \frac{1}{k!} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} + \ldots =$$

$$= \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}.$$

Kar pomeni: $\det(e^A) = \det(Q e^Z Q^T) = \det(\overbrace{Q^T Q}^{I} e^Z) = \det(e^Z) =$

$$= e^{\lambda_1} e^{\lambda_2} \cdot \ldots \cdot e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \ldots + \lambda_n} = e^{\operatorname{tr}(A)}.$$

(b) $(e^A)^T e^A \underset{\uparrow}{=} e^{A^T} e^A = e^{-A} e^A \underset{\uparrow}{=} e^{-A+A} = e^0 = I$, torej je e^A ortogonalna.

$(e^A)^T = e^{A^T}$, preveri to!

Ker $-A$ in A komutirata.

$\det(e^A) = e^{tr(A)} = e^0 = 1$.

$tr(A) = tr(A^T) = tr(-A) = -tr(A)$, torej $2 \cdot tr(A) = 0$ oz. $tr(A) = 0$.

(Pazi: V splošnem <u>ne</u> velja $e^A e^B = e^{A+B}$, velja pa, če $AB = BA$.)