Matematika 1, vaje, 17. 11. 2020

4. Naj bo N matrika

$$N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Preveri, da je množica vseh realnih 2×2 matrik, ki komutirajo z N, tj.

$$U = \{A \in \mathbb{R}^{2\times 2} : AN = NA\},$$

vektorski podprostor v $\mathbb{R}^{2\times 2}$. Poišči bazo za U in določi njegovo dimenzijo!

Preverimo, da je U res vektorski podprostor:

$A, B \in U$, tj. $AN = NA$ in $BN = NB$, tedaj

$$(\alpha A + \beta B)N = \alpha AN + \beta BN = \alpha NA + \beta NB = N(\alpha A) + N(\beta B) = N(\alpha A + \beta B),$$

tj. $\alpha A + \beta B \in U$ in U je vektorski podprostor v $\mathbb{R}^{2\times 2}$.

Kako "izgledajo" matrike iz U?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \dots \quad AN = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}, \text{ tj. } \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},$$

$$\| $$

$$NA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \quad \text{oziroma } b = 0 \text{ in } d = a.$$

Zato je $A = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$. ← tako "izgledajo" matrike iz U

To lahko zapišemo kot:

$$U \ni A = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = a\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{I} + c\overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}^{N},$$

torej $\{I, N\}$ razpenja U. Ali sta I in N linearno neodvisni? STA.

(saj $aI + cN = 0 \dots \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, tj. $a = 0$ in $c = 0 \dots$)

Torej je $B_U = \{I, N\}$ baza za U. $\dim(U) = 2$.

$(\mathcal{L}(U) = U,\ \mathcal{L}(\{I, N\}) = U \dots)$

4. Naj bo $\mathbb{R}_3[x]$ vektorski prostor polinomov p stopnje kvečjemu 3.

(a) Prepričaj se, da je preslikava $\phi\colon \mathbb{R}_3[x] \to \mathbb{R}^3$, $\phi(p) := [p(-1), p(0), p(1)]^{\mathsf{T}}$ linearna.

(b) Poišči bazo $\mathcal{B}_{\ker\phi}$ jedra $\ker\phi$ preslikave ϕ.

(c) Zapiši matriko, ki pripada ϕ v bazi $\{1, x, x^2, x^3\}$ za $\mathbb{R}_3[x]$ in standardni bazi $\mathbb{R}^3$.

(a)
$$\phi: \mathbb{R}_3[x] \longrightarrow \mathbb{R}^3, \qquad p \longmapsto \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}, \quad \text{tj. } \phi(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

S predavanj: Preslikava $\phi: \mathcal{U} \longrightarrow \mathcal{V}$ je linearna, če ohranja linearne kombinacije, tj. $\phi(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \phi(u_1) + \alpha_2 \phi(u_2)$.

$$\phi(\alpha p + \beta q) = \begin{bmatrix} (\alpha p + \beta q)(-1) \\ (\alpha p + \beta q)(0) \\ (\alpha p + \beta q)(1) \end{bmatrix} = \begin{bmatrix} \alpha p(-1) + \beta q(-1) \\ \alpha p(0) + \beta q(0) \\ \alpha p(1) + \beta q(1) \end{bmatrix} = \alpha \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \beta \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = \alpha \phi(p) + \beta \phi(q).$$

(Operaciji + in $\cdot$ sta na $\mathbb{R}_n[x]$ definirani tako: $(p+q)(x) = p(x) + q(x)$, $(\alpha p)(x) = \alpha p(x)$)

Torej je ϕ res linearna.

(b) $\ker(\phi) = \{u \in \mathcal{U} : \phi(u) = 0\} = \{p \in \mathbb{R}_3[x] : \phi(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$.

V našem primeru: $\begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, tj. $p(-1) = p(0) = p(1) = 0$ oz. p ima ničle -1, 0 in 1 (in je stopnje največ 3).

Recimo $p(x) = a(x+1)x(x-1)$ ← to so vsi polinomi iz $\ker(\phi)$.

Kaj je baza? $\mathcal{B}_{\ker\phi} = \{(x+1)x(x-1)\} = \{x^3 - x\}$. $(\dim(\ker\phi) = 1)$

(c) Matrika, ki pripada ϕ glede na bazo $\{1, x, x^2, x^3\}$ za $\mathbb{R}_3[x]$ in bazo $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ za $\mathbb{R}^3$.

$$\phi(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \vec{i} + 1 \cdot \vec{j} + 1 \cdot \vec{k} \qquad \phi(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\phi(x) \underset{\uparrow\, p(x) = x}{=} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 \cdot \vec{i} + 0 \cdot \vec{j} + 1 \cdot \vec{k} \qquad \phi(x^3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A_\phi = \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \\ \hline 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \vec{i} \\ \vec{j} \\ \vec{k} \end{array}$$

Med odmorom: Kaj je $N(A_\phi)$?

$$A_\phi \vec{x} = \vec{0}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \cancel{2}1 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 &= 0 \end{aligned} \quad \therefore \vec{x} = \begin{bmatrix} 0 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} \in N(A_\phi)$$

Glede na bazo $\{1, x, x^2, x^3\}$ so v $\vec{x}$ koeficienti linearne kombinacije baznih polinomov $\mathbb{R}_3[x]$, tj:

$$0 \cdot 1 + (-x_4)\, x + 0 \cdot x^2 + x_4 \cdot x^3 = x_4 (x^3 - x) \in \ker \phi.$$

2. Za polinom $p(x) = ax^3 + bx^2 + cx + d$ in kvadratno matriko A označimo $p(A) = aA^3 + bA^2 + cA + dI$. Naj bo $A \in \mathbb{R}^{2\times 2}$ matrika

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

(a) Prepričaj se, da je preslikava

$$\phi\colon \mathbb{R}_3[x] \to \mathbb{R}^{2\times 2}, \ \phi(p) = p(A)$$

linearna in poišči matriko, ki ji pripada v standardnih bazah prostorov $\mathbb{R}_3[x]$ in $\mathbb{R}^{2\times 2}$.

(b) Poišči bazo za $\ker \phi$ in določi $\dim(\ker \phi)$. (*Namig:* Če je $\Delta_A(\lambda)$ karakteristični polinom A, potem je $\Delta_A(A) = 0$.)

(c) Naj bo $q(x) = x(x^2 - 2x - 3)$. Ali je množica vseh 2×2 matrik X, za katere velja $q(X) = 0$, vektorski podprostor v $\mathbb{R}^{2\times 2}$?

(a) Linearnost ϕ:

$$\phi(\alpha p + \beta q) = (\alpha p + \beta q)(A) = \alpha p(A) + \beta q(A) = \alpha \phi(p) + \beta \phi(q),$$

je linearna.

$$B_{\mathbb{R}_3[x]} = \{1, x, x^2, x^3\} \qquad B_{\mathbb{R}^{2\times 2}} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$$\phi(1) = I = E_{11} + E_{22}$$

$p(x) = x$

$$\phi(x) = A = 1 \cdot E_{11} + 2 \cdot E_{12} + 2 \cdot E_{21} + 1 \cdot E_{22}$$

$p(x) = x^2$

$$\phi(x^2) = A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = 5 \cdot E_{11} + 4 \cdot E_{12} + 4 \cdot E_{21} + 5 \cdot E_{22}$$

$$\phi(x^3) = A^3 = \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix} = 13\, E_{11} + 14 \cdot E_{12} + 14 \cdot E_{21} + 13 \cdot E_{22}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\
A_\phi = & \left[\begin{array}{cccc} 1 & 1 & 5 & 13 \\ 0 & 2 & 4 & 14 \\ 0 & 2 & 4 & 14 \\ 1 & 1 & 5 & 13 \end{array}\right]
\end{array}
\begin{array}{l} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{array}
$$

(b) $\ker \phi = \{ p \in \mathbb{R}_3[x] : \underset{\phi(p)}{\underset{\parallel}{p(A)}} = \underset{\mathbb{R}^{2\times 2}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \}$

Uporabimo Cayley-Hamiltonov izrek: $\Delta_A(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, tj.

$\Delta_A \in \ker \phi$.

$$\Delta_A(x) = \det(A - xI) = \begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} = (1-x)^2 - 2^2 = x^2 - 2x - 3.$$

Torej $x^2 - 2x - 3 \in \ker \phi \subseteq \mathbb{R}_3[x]$. Polinomi st. 1 ne uničijo matrike A (saj $cA + dI \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...)

Kateri polinomi st. 3 uničijo A? Tisti, ki so deljivi z Δ_A.

Da dobimo bazo za $\ker \phi$, dodamo še $x(x^2 - 2x - 3)$.

$B_{\ker \phi} = \{ x^2 - 2x - 3, \; x(x^2 - 2x - 3) \}$, $\dim(\ker \phi) = 2$.

Ta dva sta lin. neodvisna, ker sta različnih stopenj.

(c) Ne, ni podprostor v $\mathbb{R}^{2\times 2}$.

Vemo $q(A) = 0$, vendar $q(2A) = 2A((2A)^2 - 2 \cdot 2A - 3I) =$

$$= \underbrace{2A(4A^2 - 4A - 3I)}_{\neq 0} \neq 2q(A)$$

0, tj. $2A$ ni vsebovana v množici matrik, ki jih q uniči.

1. Preslikava $\tau\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ je podana s predpisom

$$\tau(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X + X \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad , \quad K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Pokaži, da je τ linearna preslikava.

(b) Določi njeno matriko v bazi $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ prostora $\mathbb{R}^{2\times 2}$.

(a) $\tau(X) = KX + XK$

$$\tau(\alpha X + \beta Y) = K(\alpha X + \beta Y) + (\alpha X + \beta Y)K =$$

$$= \alpha KX + \beta KY + \alpha XK + \beta YK = \alpha \underbrace{(KX + XK)}_{\tau(X)} + \beta \underbrace{(KY + YK)}_{\tau(Y)},$$

tj. τ je linearna.