Matematika 1, vaje, 15. 12. 2020

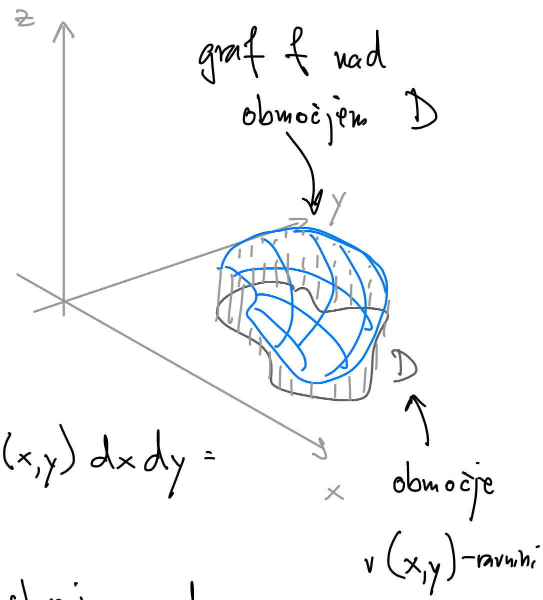
1. Izračunaj spodnje dvojne integrale.

   (a) $\iint_D (5-x-y)\,dxdy$, kjer je $D=[0,1]\times[0,1]$,

   (b) $\iint_D \frac{y}{x+1}\,dxdy$, kjer je D določeno z $x\geq 0$, $y\geq x$ in $x^2+y^2\leq 2$,

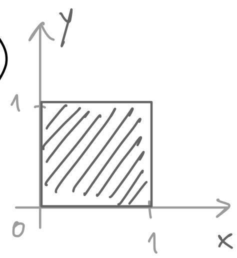
   (c) $\iint_D \frac{\sin x}{x}\,dxdy$, kjer je D trikotnik določen z $0\leq y\leq x$ in $x\leq\pi$,

   (d) $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy$ in s pomočjo tega izračunaj $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.



$$\iint_D f(x,y)\,dx\,dy = \text{„prostornina med }(x,y)\text{-ravnino in grafom } f\text{"}$$

(a)



$$\iint_D (5-x-y)\,dx\,dy = \int_0^1\left(\int_0^1(5-x-y)\,dx\right)dy = \int_0^1 dy\int_0^1(5-x-y)\,dx =$$

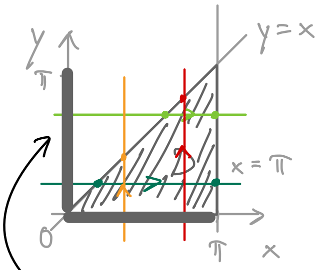
$$= \int_0^1 \left(5x-\frac{x^2}{2}-yx\right)\Big|_{x=0}^{x=1} dy = \int_0^1\left(\frac{9}{2}-y\right)dy = \left(\frac{9}{2}y-\frac{y^2}{2}\right)\Big|_{y=0}^{y=1} =$$

$$= \underline{4}.$$

$$\left(= \int_0^1 dx\int_0^1(5-x-y)\,dy = \dots = 4\right)$$

(c)

$$\iint_D \frac{\sin x}{x}\,dx\,dy =$$



projekcija D na y-os

$$= \int_0^\pi dy\int_y^\pi \frac{\sin x}{x}\,dx =$$

$$\int \frac{\sin x}{x}\,dx = \mathrm{Si}(x)+C$$

integralski sinus

= ne gre ... dobimo neelementarne integrale..

poskusimo v obratnem vrstnem redu:

$$= \int_0^\pi dx\int_0^x \frac{\sin x}{x}\,dy = \int_0^\pi\left(\frac{\sin x}{x}\cdot y\Big|_{y=0}^{y=x}\right)dx = \int_0^\pi \frac{\sin x}{\cancel{x}}\cancel{x}\,dx = (-\cos x)\Big|_{x=0}^{x=\pi} =$$

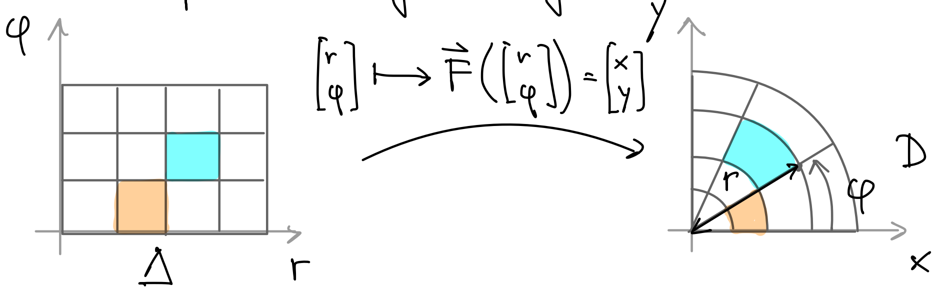
$$= -\cos\pi-(-\cos 0) = -(-1)-(-1) = \underline{2}$$

(d) $$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-x^2}\cdot e^{-y^2}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx = \int_{-\infty}^{\infty}\left(e^{-y^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) dy =$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \Big(\underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\sqrt{\pi}}\Big)^2$$

Tega bomo izračunali z uvedbo "nove" spr. v dvojni integral:



$$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(\vec{F}(r,\varphi))\cdot \det(J\vec{F}(r,\varphi))\,dr\,d\varphi$$

V našem primeru bo $\vec{F}(r,\varphi) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi\end{bmatrix} = \begin{bmatrix} x\\ y\end{bmatrix}$ (polarne koordinate)

s predavanj $\det J\vec{F}(r,\varphi) = r$ $\left(= \begin{vmatrix} x_r & y_r \\ x_\varphi & y_\varphi\end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi\end{vmatrix}\right)$

$$\underline{\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy} \underset{\uparrow}{=} \int_0^{\infty} dr \int_0^{2\pi} \underbrace{e^{-r^2\cos^2\varphi - r^2\sin^2\varphi}}_{= e^{-r^2}}\cdot r\,d\varphi =$$

uvedemo "polarne koordinate"

$$= \int_0^{\infty} dr \int_0^{2\pi} e^{-r^2}\cdot r\,d\varphi = \underline{2\pi}\int_0^{\infty} e^{-r^2}\cdot \underline{r\,dr} =$$

$\underline{2\pi}$: $\int_0^{2\pi} d\varphi$

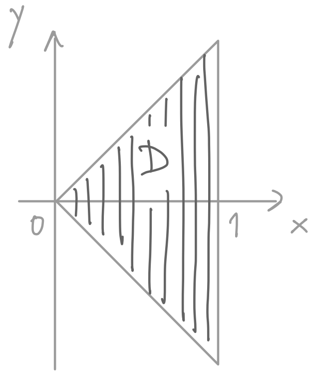
$t = -r^2 \ldots dt = -2r\,dr$

oz. $\underline{2r\,dr} = -dt$

$$= -\pi\int_0^{-\infty} e^t\,dt = -\pi\, e^t\Big|_{t=0}^{t=-\infty} = -\pi\cdot 0 - (-\pi\cdot 1) = \underline{\pi}.$$

Torej je $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

2. Skiciraj integracijsko območje in izračunaj dvakratna integrala.



(a) $\int_0^1 \left(\int_{-x}^{x} x e^y \, dy \right) dx,$

$$\int_0^1 dx \int_{-x}^{x} x e^y \, dy = \int_0^1 \left(x e^y \Big|_{y=-x}^{y=x} \right) dx =$$

$$= \int_0^1 x \left(e^x - e^{-x} \right) dx =$$

$\int u\,dv = uv - \int v\,du$

$$= x\left(e^x + e^{-x}\right)\Big|_{x=0}^{x=1} - \int_0^1 \left(e^x + e^{-x}\right) dx =$$

po delih:

$u = x$ .... $du = dx$

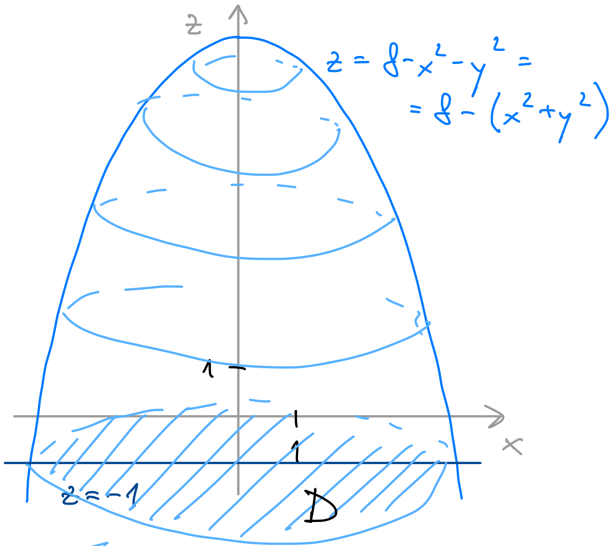
$dv = (e^x - e^{-x})\,dx$ .... $v = e^x + e^{-x}$

$$= e + \frac{1}{e} - \left(e^x - e^{-x}\right)\Big|_{x=0}^{x=1} = \cancel{e} + \frac{1}{e} - \left(\cancel{e} - \frac{1}{e}\right) = \underline{\frac{2}{e}}.$$

Poskusimo zamenjati vrstni red integracije:

$$\iint_D x e^y \, dx\,dy = \underbrace{\int_0^1 dy \int_y^1 x e^y \, dx}_{\text{le integral po delu } D \text{ nad } x\text{-osjo}} + \underbrace{\int_{-1}^{0} dy \int_{-y}^{1} x e^y \, dx}_{\text{integral po delu } D \text{ pod } x\text{-osjo}}$$

3. Izračunaj prostornino telesa, ki je omejeno s paraboloidom $z = 8 - x^2 - y^2$ in ravnino $z = -1$.



Kaj je projekcija tega telesa na (x,y)-ravnino?

$8 - x^2 - y^2 = -1$ oz. $x^2 + y^2 = 9 = 3^2$

to je krožnica s polmerom 3,

Proj. našega telesa je torej krog s polmerom 3.

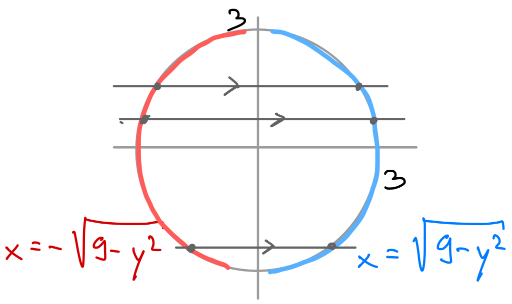
To bo hkrati naše int. območje D.

$$V = \iint_D \Big(\underbrace{(8 - x^2 - y^2) - (-1)}_{9 - x^2 - y^2}\Big)\,dx\,dy = \int_0^3 dr \int_0^{2\pi} \underbrace{(9 - r^2\cos^2\varphi - r^2\sin^2\varphi)}_{9 - r^2} \cdot \overset{\det J\vec{F}}{r}\, d\varphi =$$

uvedemo polarne koordinate, saj je D krog.

$$= 2\pi \int_0^3 (9r - r^3)\,dr = 2\pi \left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=3} = 2\pi\left(\frac{9 \cdot 3^2}{2} - \frac{3^4}{4}\right) =$$

$$= \frac{81\pi}{2}.$$



$$V = \int_{-3}^{3} dy \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (9 - x^2 - y^2)\,dx = \ldots$$