Matematika 1, vaje, 13.10.2020

1. Poišči vse lastne vrednosti in pripadajoče lastne vektorje matrike

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 1 & -3 \\ -1 & -1 & 3 \end{bmatrix}.$$

λ je **lastna vrednost** $A \in \mathbb{R}^{n\times n}$ z **lastnim vektorjem** $\vec{v} \neq \vec{0}$, če velja **$A\vec{v} = \lambda \vec{v}$.**

$$A\vec{v} - \lambda\vec{v} = \vec{0} \ \ldots \ \boxed{(A-\lambda I)\vec{v} = \vec{0}} \ \ldots \ \underbrace{\det(A-\lambda I)}_{\Delta_A(\lambda)} = 0$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 2 & 2 \\ 3 & 1-\lambda & -3 \\ -1 & -1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda-2 & 2 & 2 \\ 2+\lambda & 1-\lambda & -3 \\ 0 & -1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda-2 & 2 & 2 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} =$$

$$= (-\lambda-2)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = -(\lambda+2)\left((3-\lambda)^2 - 1\right) =$$

$$= -(\lambda+2)(3-\lambda-1)(3-\lambda+1) = -(\lambda+2)(2-\lambda)(4-\lambda) = 0$$

To so lastne vrednosti matrike A. $\rightarrow$ $\lambda_1 = -2,\ \lambda_2 = 2,\ \lambda_3 = 4$

Poiščimo še pripadajoče lastne vektorje:

- $\lambda_1 = -2$: $A - \lambda_1 I = A + 2I = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & -3 \\ -1 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x + y &= 0 \ \ldots \ x = -y \\ z &= 0 \\ 0 &= 0 \end{aligned} \qquad \vec{v}_1 = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Vzamemo $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

- $\lambda_2 = 2$: $A - \lambda_2 I = A - 2I = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -1 & -3 \\ -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

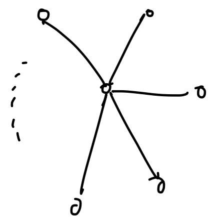
$$x - z = 0 \ \ldots \ x = z$$
$$y = 0$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \xleftarrow{\text{vzamemo}} \vec{v}_2 = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Za $\vec{v}_3$: DOMA!

3. Dana je $n \times n$ matrika

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$



tj. matrike sosednosti zvezde (kot neusmerjnega grafa).

(a) Poišči bazi za $N(A)$ in $C(A)$, tj. bazi ničelnega in stolpčnega prostora A.

(b) Poišči lastne vrednosti in lastne vektorje matrike A.
Namig: Zakaj je $N(A)$ lastni podprostor za A? Zakaj je $N(A)^\perp$ vsota ostalih lastnih podprostorov za A?

(a) $N(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$ Rešimo $A\vec{x} = \vec{0}$, $\vec{x} \in \mathbb{R}^n$ ... $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

$x_1 = 0$

$x_2 + x_3 + \dots + x_n = 0$ ... $x_2 = -(x_3 + x_4 + \dots + x_n)$.

$$\vec{x} = \begin{bmatrix} 0 \\ -(x_3 + x_4 + \dots + x_n) \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_4 \\ 0 \\ x_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ -x_n \\ 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Torej je: $B_{N(A)} = \Bigg\{ \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{n-2} \Bigg\}$, $\dim N(A) = n-2$.

$C(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ .... $A\vec{x} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$

$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n$

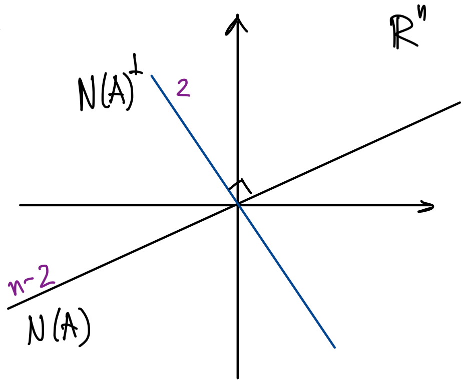
$B_{C(A)} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right\}$ ... $\dim C(A) = 2$

(b) $A\vec{x} = \lambda\vec{x}$ $\qquad$ $A\vec{x} = \vec{0} = 0\cdot\vec{x}$, tj. vsak $\vec{x} \in N(A)$ je lastni vektor A za lastno vrednost 0.

Torej A ima $(n-2)$-kratno lastno vrednost 0, pripadajoči linearno neodvisni lastni vektorji so iz $B_{N(A)}$.

Matrika $A \in \mathbb{R}^{n\times n}$ ima največ n linearno neodvisnih lastnih vektorjev. Ker je naša A simetrična, ima še točno 2 lin. neodv. l. v..
Lastni vektorji za različne lastne vrednosti sim. mat. A so avtomatično pravokotni med sabo.

Lastni vekt. za $\lambda \neq 0$ so pravokotni na vse l. vekt. za $\lambda = 0$, tj. na $B_{N(A)}$, to so vektorji iz $N(A)^\perp$:



Iščemo: $\vec{y}\cdot\begin{bmatrix}0\\-1\\1\\0\\ \vdots\\0\end{bmatrix} = 0$, $\vec{y}\cdot\begin{bmatrix}0\\-1\\0\\1\\0\\ \vdots\\0\end{bmatrix} = 0$, $\ldots$, $\vec{y}\cdot\begin{bmatrix}0\\-1\\0\\ \vdots\\0\\1\end{bmatrix} = 0$.

$\vec{y} = \begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$: $-y_2 + y_3 = 0$, $-y_2 + y_4 = 0$, $\ldots$, $-y_2 + y_n = 0$

$y_3 = y_2$, $y_4 = y_2$, $\ldots$, $y_n = y_2$

Torej $\vec{y} \in N(A)^\perp$ je oblike $\vec{y} = \begin{bmatrix}y_1\\y_2\\y_2\\ \vdots\\y_2\end{bmatrix}$.

Poiščimo še ostale l. v. in l. v. direktno iz def. $(A\vec{y} = \lambda\vec{y})$:

$$A\vec{y} = \begin{bmatrix}0 & 1 & \cdots & 1\\ 1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_2\\ \vdots\\y_2\end{bmatrix} = \begin{bmatrix}(n-1)y_2\\ y_1\\ y_1\\ \vdots\\ y_1\end{bmatrix} = \lambda\vec{y} = \begin{bmatrix}\lambda y_1\\ \lambda y_2\\ \vdots\\ \lambda y_2\end{bmatrix} \ \ldots \ \begin{aligned}(n-1)y_2 &= \lambda y_1 = \lambda^2 y_2\\ y_1 &= \lambda y_2\end{aligned}$$

$\ldots \ (n-1)y_2 = \lambda^2 y_2 \ \ldots \ (n-1) = \lambda^2 \ \ldots \ \lambda = \pm\sqrt{n-1}$.

Upoštevamo še $y_1 = \lambda y_2 = \pm\sqrt{n-1}\, y_2$.

Končno: $\lambda = \pm\sqrt{n-1}$ pripada l. vektor $\vec{y} = \begin{bmatrix}\pm\sqrt{n-1}\, y_2\\ y_2\\ \vdots\\ y_2\end{bmatrix} = y_2\begin{bmatrix}\pm\sqrt{n-1}\\ 1\\ 1\\ \vdots\\ 1\end{bmatrix}$.

4. O *simetrični* matriki $A \in \mathbb{R}^{4\times 4}$ vemo naslednje: 3 je 2-kratna lastna vrednost A, vektorja

$$\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \text{ in } \mathbf{v}_2 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

pa A slika enega v drugega, tj. $A\mathbf{v}_1 = \mathbf{v}_2$ in $A\mathbf{v}_2 = \mathbf{v}_1$. Poišči tako matriko A ali pa utemelji, zakaj ne obstaja!

$$\left.\begin{array}{l} A\vec{v}_1 = \vec{v}_2 \\ A\vec{v}_2 = \vec{v}_1 \end{array}\right\} \begin{array}{l} + \rightsquigarrow A\vec{v}_1 + A\vec{v}_2 = \vec{v}_2 + \vec{v}_1 \quad \ldots \quad A(\vec{v}_1 + \vec{v}_2) = \vec{v}_1 + \vec{v}_2 . \\ - \rightsquigarrow A\vec{v}_1 - A\vec{v}_2 = \vec{v}_2 - \vec{v}_1 \quad \ldots \quad A(\vec{v}_1 - \vec{v}_2) = -(\vec{v}_1 - \vec{v}_2) . \end{array}$$

Torej: $\vec{v}_1 + \vec{v}_2$ je lastni vektor A za l. vrednost 1, $\quad \vec{v}_1 + \vec{v}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} =: \vec{u}_1$

$\vec{v}_1 - \vec{v}_2$ je lastni vektor A za l. vrednost -1. $\quad \vec{v}_1 - \vec{v}_2 = \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} =: \vec{u}_2$

Lastna vektorja, ki pripadata (2-kratni) l. vred. 3 sta (ker je A sim.) pravokotna na $\vec{u}_1$ in $\vec{u}_2$.

$\vec{u}_1 \cdot \vec{u} = 0$, $\quad$ pišimo $\vec{u} = \begin{bmatrix}x\\y\\z\\w\end{bmatrix}$, dobimo $\quad x + y + z + w = 0$

$\vec{u}_2 \cdot \vec{u} = 0 \qquad\qquad\qquad\qquad\qquad -x + y + z - w = 0$

torej $2y + 2z = 0$, tj. $y = -z$,

in $2x + 2w = 0$, tj. $x = -w$. Rešitev $\vec{u} = \begin{bmatrix}-w\\-z\\z\\w\end{bmatrix} = z\underbrace{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}}_{=\vec{u}_3} + w\underbrace{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}_{=\vec{u}_4}$

linearno neodvisna last. vektorja A za lastno vrednost 3 $\rightarrow$ $\vec{u}_3$, $\vec{u}_4$

A je torej podobna diagonalni matriki

$$D = \begin{bmatrix}-1 & & & \\ & 1 & & \\ & & 3 & \\ & & & 3\end{bmatrix} \quad \text{s prehodno matriko} \quad P = [\vec{u}_2, \vec{u}_1, \vec{u}_3, \vec{u}_4] = \begin{bmatrix}-1 & 1 & 0 & -1\\ 1 & 1 & -1 & 0\\ 1 & 1 & 1 & 0\\ -1 & 1 & 0 & 1\end{bmatrix},$$

tj. $A = PDP^{-1}$.