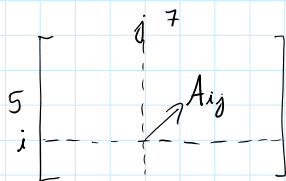


$\mathbb{R}^{n \times n} \rightarrow n$ vrstic, n stolpcev & realni elementi



obrazloži se linearna odvisnost vrstic

RANG MATRIKE

$\rightarrow A \xrightarrow{\text{GAUSS ELIM.}} \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \rightarrow \text{rang } A$

$$\begin{bmatrix} 3 & 0 & -6 \\ 3 & -3 & -3 \\ 2 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad 2$$

$\rightarrow$ število linearno neodvisnih vrstic/stolpcev matrice

NIČELNI PROSTOR MATRIKE

$$N(A) = \{ \vec{x} \in \mathbb{R}^n; A\vec{x} = \vec{0} \} \in \mathbb{R}^n \quad \vec{x} = \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \mathcal{L} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{aligned} x_1 - 2x_3 &= 0 & x_1 &= 2x_3 \\ x_2 - x_3 &= 0 & x_2 &= x_3 \\ 0 &= 0 \end{aligned} \rightarrow x_1 = 2x_3, x_2 = x_3$$

PROSTA SPREMENLJIVKA

STOLPČNI PROSTOR MATRIKE

$$C(A) = \mathcal{L} \{ A^{(1)}, A^{(2)}, \dots, A^{(n)} \} \in \mathbb{R}^n$$

$\downarrow$ STOLPEC

$$\dim N(A) + \dim C(A) = n \quad \text{in} \quad N(A) \perp C(A)$$

LASTNE VREDNOSTI MATRIK

$A \in \mathbb{R}^{n \times n}$ za $\vec{x} \in \mathbb{R}^n$ velja $A\vec{x} = \lambda\vec{x}$

LASTNI VEKTOR $\uparrow$ LASTNA VREDNOST $\uparrow$

$$\begin{aligned} A\vec{x} - \lambda I\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

$$\vec{x} \in N(A - \lambda I) \Leftrightarrow A - \lambda I \text{ ni obrnljiva}$$

$$\Leftrightarrow \det(A - \lambda I) = 0$$

SLED MATRIKE

Za matriko $\mathbb{R}^{n \times n}$ definiramo njeno sled kot $\text{tr}(A) = \sum_{i=1}^n A_{ii} \in \mathbb{R}$.

LASTNOSTI: 1) $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ dokaz: $\text{tr}(A+B) = \sum_{i=1}^n (A+B)_{ii} = \sum_{i=1}^n (A_{ii} + B_{ii}) = \sum_{i=1}^n A_{ii} + \sum_{i=1}^n B_{ii} = \text{tr}(A) + \text{tr}(B)$

2) $\text{tr}(dA) = d \text{tr}(A)$

3) $\text{tr}(AB) \neq \text{tr}(A)\text{tr}(B)$

4) $\text{tr}(AB) = \text{tr}(BA)$ $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}$

5) $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$ dokaz 4): $\text{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ji} = \sum_{j=1}^n \sum_{i=1}^n B_{ji} A_{ij} = \sum_{j=1}^n (BA)_{jj} = \text{tr}(BA)$

6) $\text{tr}(PBP^{-1}) = \text{tr}(P^{-1}PB) = \text{tr}(B)$

PODOBNI MATRIKI

Matrika $A \in \mathbb{R}^{n \times n}$ in $B \in \mathbb{R}^{n \times n}$ sta podobni, če obstaja obrnljiva matrika $P \in \mathbb{R}^{n \times n}$, da velja $A = PBP^{-1}$.

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$$

$\leftarrow$ PODOBNI

Podobne matrike imajo: isti rang (množica & obrnljiva matrika ohranjata rang)
- isto sled

Podobne matrice imaju: - isti rang (množica je odnosa matrica odnosa rang)

- isto sled

- isti karakteristični polinom

- isto lastne vrednosti

$$\Delta_A(x) = \det(A - xI) \quad \text{dokaz: } A = PBP^{-1}$$

$$\begin{aligned} \Delta_A(x) &= \det(A - xI) \\ &= \det(PBP^{-1} - xI) \\ &= \det(PBP^{-1} - P(xI)P^{-1}) \\ &= \det(P(B - xI)P^{-1}) \\ &= \det(P) \det(B - xI) \det(P^{-1}) \\ &= \det(B - xI) \underbrace{\det(P) \det(P^{-1})}_{\det(PP^{-1}) = \det(I) = 1} \\ &= \Delta_B(x) \end{aligned}$$

POZITIVNA/NEGATIVNA (SEMI) DEFINITNOST

DOSLED: $\begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x-2y \\ 3y \end{bmatrix} \leftarrow \text{LINEARNA FORMA}$
 $\leftarrow \text{LINEARNA FORMA}$

DANES:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3x-2y \\ 3y \end{bmatrix} = 3x^2 - 2xy + 3y^2 \leftarrow \text{KVAORATNA FORMA}$$

$$\begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

VEČ MATRIK, SIMETRIČNE

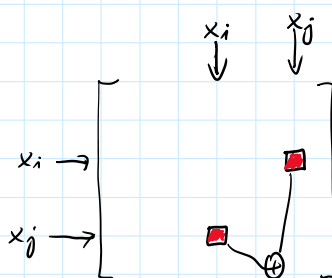
SO NAM NAJLJUBŠE

ZARADI DIAAGONALIZACIJE

Za simetrično matriko $M \in \mathbb{R}^{n \times n}$ je izraz oblike $\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ KVAORATNA NORMA.

KVAORATNA FORMA: $\vec{x}^T M \vec{x}$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + (b+c)xy + dy^2$$



$$3x^2 - 2xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

SIMETRIČNA

$$\vec{x}^T M \vec{x} = \vec{x}^T Q D Q^T \vec{x} = \vec{y}^T D \vec{y} \leftarrow \text{SAMO KVAORATNI ČLENI M I MEŠANIH ČLENOV}$$

ORIGINALNE SPREMEMNIVKE $\uparrow$ $\vec{y} = Q^T \vec{x}$ $\uparrow$ NOVE SPREMEMNIVKE

$M = Q D Q^T$
 Q je ortogonalna
 $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
 LINEARNA ZAMENJAVA SPREMEMNIVK

Našino matriko tako rekoč imamo $3x^2 - 2xy + 3y^2 = 4$

$$M = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \xrightarrow{\text{LASTNE VREDNOSTI}} \lambda_1 + \lambda_2 = \text{tr}(M) = 6 \quad \lambda_1 = 4$$

$$\lambda_1 \lambda_2 = \det(M) = 8 \quad \lambda_2 = 2$$

$$\lambda_1 = 4 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x+y=0 \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x-y=0 \quad \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow 4x^2 + 2y^2 = 4$$

NOVE:

$$\vec{y} = \begin{bmatrix} x \\ y \end{bmatrix}$$

STARJE:

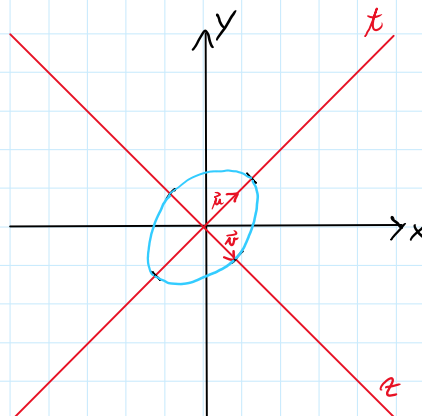
$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

LEGA NOVE

$$Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Q^{(1)}$$

$$x + \frac{y}{\sqrt{2}} = 1$$

$$x + \frac{y}{\sqrt{2}} = 1$$



NOVE:

$$\vec{y} = \begin{bmatrix} z \\ t \end{bmatrix}$$

STARŠ:

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

LEGA NOVE
Z-OSI

$$Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Q^{(1)}$$

$$z + \frac{t}{\sqrt{2}} = 1$$

$$z + \frac{t}{\sqrt{2}} = 1$$

LEGA NOVE
T-OSI

$$Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} = Q^{(2)}$$

Za simetrično matriko $M \in \mathbb{R}^{n \times n}$ pravimo da je

a) POZITIVNO DEFINITNA (PD), če $\vec{x}^T M \vec{x} \geq 0$ za vse $\vec{x} \in \mathbb{R}^n$

$$\text{in } \vec{x}^T M \vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$$

b) POZITIVNO SEMI-DEFINITNA (PSD), če $\vec{x}^T M \vec{x} \geq 0$ za vse $\vec{x} \in \mathbb{R}^n$

VSAKA PD JE TUDI PSD
 $PD \Rightarrow PSD$

$$\text{PRIMER: } \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

je PD, ker je njena kvadratična forma enaka $4x^2 + 2y^2 \geq 0$ in $= 0 \Leftrightarrow x = y = 0$.

KOVARIANČNA matrika je PSD

GRAMOVA matrika je PSD

jedna SVD so PSD

$$x^2 - y^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$x^2 - y^2 \Big|_{x=1, y=0} = 1 \quad x^2 - y^2 \Big|_{x=0, y=1} = -1 \quad x^2 - y^2 \Big|_{x=0, y=0} = 0$$

NEDEFINITNA

$$x^2 - 2xy + y^2 = (x - y)^2 \text{ je PSD ni pa PD}$$

c) NEGATIVNO DEFINITNA (ND), če $\vec{x}^T M \vec{x} \leq 0$ za vse $\vec{x} \in \mathbb{R}^n$

$$\text{in } \vec{x}^T M \vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$$

d) NEGATIVNO SEMI-DEFINITNA (NSD), če $\vec{x}^T M \vec{x} \leq 0$ za vse $\vec{x} \in \mathbb{R}^n$

e) NEDEFINITIVNA, če $\vec{x}^T M \vec{x} > 0$ za nek $\vec{x} \in \mathbb{R}^n$ in $\vec{y}^T M \vec{y} < 0$ za nek $\vec{y} \in \mathbb{R}^n$.

IZREK:

Matrika M je PD natanko tedaj, ko so vse njene lastne vrednosti pozitivne > 0

-||-

PSD

-||-

nonnegative ≥ 0

-||-

ND

-||-

negative < 0

-||-

NSD

-||-

nonpositive ≤ 0

-||-

nedefinitna ko ima najmanj eno pozitivno in najmanj eno negativno lastno vrednost.

DOKAZ ZA PSD:

$$M = Q D Q^T$$

$$\vec{x}^T M \vec{x} = \vec{x}^T Q D Q^T \vec{x} = \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

$(\Rightarrow)$ M PSD

$$\vec{x}^T M \vec{x} \geq 0 \Rightarrow \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 \geq 0$$

$$\text{Če } \lambda_i < 0, \text{ potem je } \lambda_i \cdot 0 + \dots + \lambda_i \cdot 1 + \dots + \lambda_n \cdot 0 < 0 \quad \text{---}$$

Gledaj $\lambda_i \geq 0$

$$(\Leftarrow) \text{ Če } \lambda_1, \dots, \lambda_n \geq 0, \text{ potem } \underbrace{\vec{x}^T M \vec{x} = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2}_{\substack{0 \leq \lambda_i \\ 0 \leq y_i^2}} \geq 0 \Rightarrow M \text{ je PSD}$$

$A \in \mathbb{R}^{m \times m}$ in $B \in \mathbb{R}^{n \times n}$ PSD $\Rightarrow A \otimes B \in \mathbb{R}^{(m \cdot n) \times (m \cdot n)}$ je PSD.

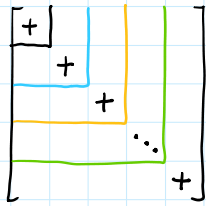
A ima l. vrednosti $\lambda_1, \dots, \lambda_m \geq 0$

B ima l. vrednosti $\mu_1, \dots, \mu_n \geq 0$

$A \otimes B$ ima lastne vrednosti $\{\lambda_i \mu_j; i=1, \dots, m, j=1, \dots, n\} \geq 0 \Rightarrow A \otimes B \text{ je PSD.}$

M je PD $\rightarrow$ nobena njena "lani njegovi rogal" je PD

M je PD $\rightarrow$ enak je "levo simetričen" je PD



SILVESTROV IZREK:

M je PD $\Leftrightarrow \det(M_{(k)}) > 0$ za vsa $k=1 \dots n$
 $M_{(k)}$ "levo simetričen" $k \times k$ podmatrica M

M je ND $\Leftrightarrow \det(M_{(k)}) = \begin{cases} > 0 & \text{če } k \text{ sod} \\ < 0 & \text{če } k \text{ lič} \end{cases}$

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad \begin{matrix} 3 > 0 \\ 3 \cdot 3 - 1 = 8 > 0 \end{matrix} \Rightarrow \text{PD}$$

CHOLESKY IZREK:

$M \in \mathbb{R}^{n \times n}$ je simetričen PD $\Leftrightarrow M = LL^T$ za neko **spodnje** Δ matriko.

$$\begin{bmatrix} & & \\ & 0 & \\ * & & \end{bmatrix} = \begin{bmatrix} * & & \\ & 0 & \\ & & *^T \end{bmatrix}$$

PRODUKT DVEH OBRNLJIVIH
JE OBRNLJIVA

DOKAZ:

$\Leftarrow$ Če $M = LL^T$, potem je
 - simetričen (ker $M^T = (LL^T)^T = (L^T)^T L^T = LL^T$)
 - PD ($\vec{x}^T M \vec{x} = \vec{x}^T L L^T \vec{x} = (L \vec{x})^T L \vec{x} = \|L \vec{x}\|^2 \geq 0$ M je PSD + obrnljiva $\rightarrow$ PD)

$\Rightarrow$ M simetričen + PD

$$M = Q D Q^T$$

ORTOGONALNA

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \lambda_i > 0$$

$$E = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

KORENIMO LAHKO
KER PD

$$M = Q D Q^T = Q E E^T Q^T = (Q E) (Q E)^T = B B^T$$

OBRNLJIVA
OBRNLJIVA
NI SPODNJE Δ

$$B^T = Q^T R \quad \text{je QR razcep } B^T$$

$$M = B B^T = (Q^T R)^T Q^T R = R^T Q^{TT} Q^T R = R^T R$$

ORTOGONALNA Δ I Δ
ZGORNJE Δ

$$L = R^T$$

↓
SKALARJI

elementi: $\mathbb{R}$ $\mathbb{R}^n$ $\mathbb{R}^{n \times n}$ $\rightarrow$ funzione

(Abstraktni realni) vektorski prostor V je množica elementov $v \in V$, ki jih imenujemo vektorji, skupaj z dveh notranjima operacijama:

- stetig ($u, v \in V \Rightarrow u+v \in V$)
- linear $\approx$ skalieren ($v \in V, d \in \mathbb{R} \Rightarrow d \cdot v \in V$)

in 2 parts:

- 1) $(u+v)+w = u+(v+w)$ ASOCIATIVNOST
- 2) $u \cdot v = v \cdot u$ KOMUTATIVNOST
- 3) obstajajo $0_V \in V$, da velja $v+0_V = 0_V+v = v$ (ničelni element, enota)
- 4) za vsak $v \in V$ obstaja nasprotni element $-v$, da velja $v+(-v) = (-v)+v = 0_V$
- 5) $1 \cdot v = v$
- 6) $d(\beta v) = \beta(dv) = (d\beta)v$
- 7) $d(u+v) = du+dv$
- 8) $(d+\beta)v = dv+\beta v$ } DISTRIBUTIVNOST

Sei nun $u, v, w \in V$ in $\alpha, \beta \in \mathbb{R}$

V vektorski prostor in $U \subseteq V$ podmnožica:

- toreži: prostor in $U \subseteq V$ podprostor:
 - jame, da velja 1, 2, 5, 6, 7, 8 tudi so ne $u \in U$
 - je +, - notranji v U , potem jame $0 \in U$
 - -11-, potem so $u \in U$ je tudi $(-u) \in U$
- $$\begin{aligned}
 &u \in U \quad 0 \in \mathbb{R} \\
 &0u = (0+0)u = 0u + 0u \quad /+(-0u) \\
 &0v = 0u + 0v \\
 &0u = 0v
 \end{aligned}$$

Če je dana matrika U v vektorskem prostoru V , lahko preverimo, da je U !!!
tudi vektorski prostor, da preverimo SAMO, da sta $+$ in $\cdot$ notranji operaciji. !!!

U je vektorski podprostor w V , to

- $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$
- $\lambda \in R, u \in U \Rightarrow \lambda u \in U$

Ali je množica $S_n = \{A \in \mathbb{R}^{n \times n}; A = A^T\}$ vektorski podprostor v $\mathbb{R}^{n \times n}$?

- ① $A, B \in \mathcal{Y}_n \Rightarrow A+B \in \mathcal{Y}_n$
- ② $A \in \mathcal{Y}_n, \lambda \in \mathbb{R} \Rightarrow \lambda A \in \mathcal{Y}_n$

$$\textcircled{2} \tilde{A} \in \tilde{\mathcal{L}}_n, \tilde{\alpha} \in \mathbb{R} \Rightarrow \tilde{\alpha} \tilde{A} \in \tilde{\mathcal{L}}_n$$

$$\textcircled{1} A^T = A \text{ in } B^T = B \quad (A+B)^T = A+B ??$$

$$(A+B)^T = A^T + B^T = A+B$$

$$\textcircled{2} A^T = A \quad (\alpha A)^T = \alpha A ??$$

$$(\alpha A)^T = \alpha A^T = \alpha A$$

EKVIVALENTNO: U je podprostor $\Leftrightarrow \forall u_1, u_2 \in U$ in $\alpha_1, \alpha_2 \in \mathbb{R}$ velja $\alpha_1 u_1 + \alpha_2 u_2 \in U$. $\Leftrightarrow$

$$\begin{aligned} u_1 + u_2 &= 1u_1 + 1u_2 \\ \alpha u_1 &= \alpha u_1 + 0u_2 \end{aligned}$$

$$(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$$

Za elemente $v_1, \dots, v_k \in V$ imenujemo
 $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$
 linearna kombinacija vektov $v_1, \dots, v_k$.

Množica vseh linearnih kombinacij imenujemo LINEARNA OGRANJACA (LUPINA, SPAN).

$$\mathcal{L} = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k; \alpha_1, \dots, \alpha_k \in \mathbb{R} \}$$

Dan je vektor $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$. Ustanovi $U = \{ X \in \mathbb{R}^{2 \times 2}; \vec{a}^T X \vec{a} = 0 \}$. Ali je U vektorski prostor?

$U \subseteq \mathbb{R}^{2 \times 2}$ in $\mathbb{R}^{2 \times 2}$ je vektorski prostor.

$$\left. \begin{aligned} 1) X, Y \in U &\Rightarrow \vec{a}^T X \vec{a} = 0 \text{ in } \vec{a}^T Y \vec{a} = 0 \\ &\vec{a}^T (X+Y) \vec{a} = \vec{a}^T X \vec{a} + \vec{a}^T Y \vec{a} = 0+0=0 \\ 2) X \in U, \alpha \in \mathbb{R} &\Rightarrow \vec{a}^T X \vec{a} = 0 \\ &\vec{a}^T (\alpha X) \vec{a} = \alpha \vec{a}^T X \vec{a} = \alpha 0 = 0 \end{aligned} \right\} U \text{ je vektorski prostor} \leftarrow \text{ABSTRAKTEN DOKAZ}$$

Kakor izgledajo elementi U ?

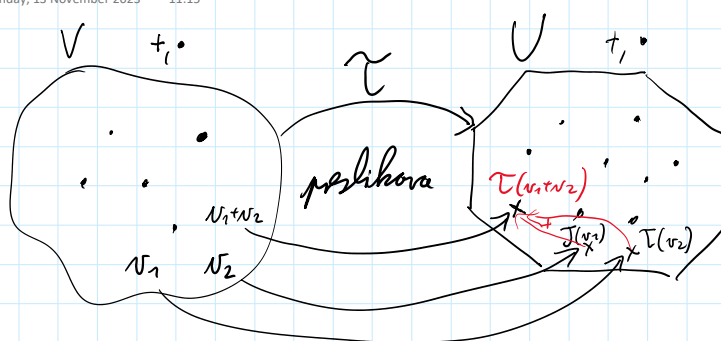
$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} + 2x_{12} \\ x_{21} + 2x_{22} \end{bmatrix} = x_{11} + 2x_{12} + 2x_{21} + 4x_{22} = 0$$

$$x_{11} = -2(x_{12} + x_{21} + 2x_{22})$$

$$\begin{aligned} U &= \left\{ X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}; x_{11} = -2(x_{12} + x_{21} + 2x_{22}) \right\} \\ &= \left\{ X = \begin{bmatrix} -2x_{12} - 2x_{21} - 4x_{22} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}; x_{12}, x_{21}, x_{22} \in \mathbb{R} \right\} \end{aligned}$$

KONKRETEN DOKAZ



Naj bosta V in U nekateri prostora. Preslikava $T: V \rightarrow U$ je LINEARNA, če:

$$1) T(v_1 + v_2) = T(v_1) + T(v_2) \quad \text{ADITIVNOST}$$

$$2) T(\alpha v_1) = \alpha T(v_1)$$

za poljubni $v_1, v_2 \in V$ in $\alpha \in \mathbb{R}$.

NULNI VEKTOR SE SLIKA V NULNI VEKTOR.
M PA DOVOLJ!!!

PRIMERI LINEARNIH:

$$1. T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{— ni injektivna}$$

$$T \text{ je linearna: } T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \quad \checkmark$$

$$T(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} = \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) \quad \checkmark$$

$$2. \varphi: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \quad \varphi(A) = A^T$$

$$\varphi(A+B) = (A+B)^T = A^T + B^T = \varphi(A) + \varphi(B)$$

$$\varphi(\alpha A) = (\alpha A)^T = \alpha A^T = \alpha \varphi(A)$$

$$3. T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ je linearna, vendar ni injektivna} \Rightarrow \text{ni izobitna}$$

PRIMERI NELINEARNIH:

$$1) J_1: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$J_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$T_1\left(2\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

$$2 T_1\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \quad \text{ne enako}$$

$$2) \pi: \mathbb{R} \rightarrow \mathbb{R}$$

$$\pi(x) = x + \pi \quad \pi(0) = 0 + \pi \neq 0$$

Linearna preslikava $\mathbb{R} \rightarrow \mathbb{R}$
so $T(x) = kx + \underline{0}$

$$\pi(1+2) = \pi(3) = 3 + \pi$$

$$\pi(1) + \pi(2) = 1 + \pi + 2 + \pi = 3 + 2\pi$$

$$3) k: \mathbb{R} \rightarrow \mathbb{R}$$

$$k(x) = x^2$$

$$k(0) = 0^2 = 0, \text{ a NI LINEARNA}$$

$$k(1+2) = k(3) = 9$$

$$k(1) + k(2) = 1 + 4 = 5 \quad \text{ne enako}$$

NAJPOMEMBNEJSI PRIMER:

$$A \in \mathbb{R}^{n \times n} \quad \varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\varphi(\vec{x}) = A\vec{x}$$

- 1) $\varphi(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \varphi(\vec{x}) + \varphi(\vec{y})$ ✓✓
- 2) $\varphi(d\vec{x}) = A(d\vec{x}) = d(A\vec{x}) = d\varphi(\vec{x})$

MATRIKE, KI PRIPADAJO LINEARNIM PRESLIKAVAM

V, U vektorski prostora in $\tau: V \rightarrow U$ linearna preslikava
 $B = \{v_1, \dots, v_m\}$ baza V . Če poznamo $\tau(v_1) \dots \tau(v_m)$, potem je
 linearna preslikava τ **uniqueno določena**.

Naj $w \in U$: $w = d_1 v_1 + \dots + d_m v_m$ so skalarne določene $d_i \in \mathbb{R}$
 $\tau(w) = \tau(d_1 v_1 + \dots + d_m v_m) = d_1 \tau(v_1) + d_2 \tau(v_2) + \dots + d_m \tau(v_m)$
 τ LINEARNA

KOEFICIENTI PRI RAZVOJU
 $\tau(v_i)$ po bazi C v U

$$C = \{u_1, \dots, u_n\} \text{ baza } U$$

$$\Rightarrow \tau(v_1) = \beta_{11} u_1 + \beta_{21} u_2 + \dots + \beta_{n1} u_n$$

$$\tau(v_2) = \beta_{12} u_1 + \beta_{22} u_2 + \dots + \beta_{n2} u_n$$

$$\vdots$$

$$\tau(v_m) = \beta_{1m} u_1 + \beta_{2m} u_2 + \dots + \beta_{nm} u_n$$

$$A_{\tau, B, C} = \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

V KATERO BAZO
 PROJEKCIJA
 IZ KATERE BAZE

PRIMER:

- 1) $\tau\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ Določimo matriko, ki pripada τ je baza
 $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ v baza $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

V prostoru $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$

$\mathcal{P}_n = \{e_1, \dots, e_n\}$ imenujemo **STANDARDNA**
 baza $\mathbb{R}^n$.

- 1) $\tau\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
 $\tau\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2) Razvijem po bazi C

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_{\tau, B, C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\varphi: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}_2[x]$$

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-b)x^2 + (b-c)x + (c-d)$$

a) Pokazati, da je linearna D.N.

b) Zapišimo matriko, ki pripada τ iz standardne baze $\mathbb{R}^{2 \times 2}$ v standardno bazo $\mathbb{R}_2[x]$

$$A_{\tau, \underbrace{\mathcal{S}_{\mathbb{R}^{2 \times 2}}}_{\dim=4}, \underbrace{\mathcal{S}_{\mathbb{R}_2[x]}}_{\dim=3}} = \begin{bmatrix} \varphi(e_{11}) & \varphi(e_{12}) & \varphi(e_{21}) & \varphi(e_{22}) \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

$\mathcal{S}_{\mathbb{R}^{2 \times 2}} = \{1, x, x^2\}$, to bi realni $\{x^2, x, 1\}$,
 bi v matriki rangirali 1. in 3. vrstico.

$$A_{\tau, \underbrace{S_{E_{12}}}_{\text{STOLPICI}}, \underbrace{S_{E_{12}}}_{\text{VRSTICE}}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{matrix} x \\ x^2 \end{matrix}$$

$\tau(1, x, x^2, 1)$ in $\tau(1, x, x^2, 1)$
 li n matriki rangajbo 1. in 3. vrstice.

$$\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1x^2 + 0x + 0$$

$$\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -1x^2 + 1x + 0$$

$$\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0x^2 - 1x + 1$$

$$\varphi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0x^2 - 0x - 1$$

POČUNANJE
MATEMATIKA

$$\begin{aligned} \varphi(B) &= \varphi(d_1 E_{11} + d_2 E_{12} + d_3 E_{21} + d_4 E_{22}) \\ &= d_1 \varphi(E_{11}) + d_2 \varphi(E_{12}) + d_3 \varphi(E_{21}) + d_4 \varphi(E_{22}) \\ &= d_1 (\beta_{11} 1 + \beta_{21} x + \beta_{31} x^2) + \\ &\quad d_2 (\beta_{12} 1 + \beta_{22} x + \beta_{32} x^2) + \\ &\quad \vdots \\ &\quad d_4 (\beta_{14} 1 + \beta_{24} x + \beta_{34} x^2) \\ &= (d_1 \beta_{11} + d_2 \beta_{12} + d_3 \beta_{13} + d_4 \beta_{14}) \cdot 1 + \\ &\quad (d_1 \beta_{21} + d_2 \beta_{22} + d_3 \beta_{23} + d_4 \beta_{24}) \cdot x + \\ &\quad (d_1 \beta_{31} + d_2 \beta_{32} + d_3 \beta_{33} + d_4 \beta_{34}) \cdot x^2 \\ &= \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \\ &= A_{\tau, B, C} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \text{koeficienti pred } 1, x, x^2 \text{ tj. } \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix} \end{aligned}$$

VEKTORSKI PROSTOR

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

KOEFICIENTI

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ po } \text{bazi } \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$$\varphi \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

$-1 - 1x - 1x^2$

JEDRO IN SLIKA LINEARNE PRESLIKAVE

Jedro linearne preslikave $\tau: V \rightarrow V$ je množica $\ker(\tau) = \{v \in V; \tau(v) = 0_v\}$

- $0_v \in \ker(\tau)$
- $\ker(\tau)$ je vektorski podprostor n V DN
- τ je injektivna $\Leftrightarrow \ker(\tau) = \{0_v\}$
 - $\nexists v_1 \neq v_2$ tj. tudi $\tau(v_1) \neq \tau(v_2)$
 - $\Leftrightarrow \tau(v_1) = \tau(v_2) \Rightarrow v_1 = v_2$

$(\Rightarrow)$ τ je injektivna tudi li pokazati, da je jedro trivialno
 $\Leftrightarrow v \in \ker(\tau) \Rightarrow \tau(v) = 0_v = \tau(0_v)$
 $v = 0_v$ ✓

$(\Leftarrow)$ $\ker(\tau) = \{0_v\}$. Radi li pokazati, da iz $\tau(v_1) = \tau(v_2)$ sledi $v_1 = v_2$.

$$\begin{aligned} \Leftrightarrow \tau(v_1) &= \tau(v_2) \\ \tau(v_1) - \tau(v_2) &= 0_v \downarrow \text{LINEARNOST} \\ \tau(v_1 - v_2) &= 0_v \\ v_1 - v_2 &\in \ker(\tau) = 0_v \\ v_1 - v_2 &= 0_v \end{aligned}$$

$$\begin{aligned} |n_1 - n_2| &= 0 \\ n_1 - n_2 &\in \mathcal{L}(\mathcal{V}) = 0 \\ n_1 - n_2 &= 0 \\ \underline{n_1 = n_2} \end{aligned}$$

- $v \in \ker(\tau) \quad \tau(v) = 0_v$

$$A = A_{T, B, C}$$

$$v = d_1 v_1 + \dots + d_m v_m \text{ rozvoj po } B = \{v_1, \dots, v_m\}$$

$$\tau(w)=0 \Leftrightarrow A \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \vec{0}$$

$$\Leftrightarrow \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \in N(A)$$

- $\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-b)x^2 + (b-c)x + (c-d)$

$$\ker(\varphi) = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \underbrace{(a-b)}_{=0} x^2 + \underbrace{(b-c)}_{=0} x + \underbrace{(c-d)}_{=0} = 0 \text{ for all } x \in \mathbb{R}\}$$

$$= \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a=b=c=d \}$$

$$= \{ \begin{bmatrix} a & a \\ a & a \end{bmatrix}; a \in \mathbb{R} \}$$

$$= \mathcal{L}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

$$A_{\varphi, \varphi_{2v_2}, \varphi_{2v_5}} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$N(A_1, g_{2 \times 2}, \varphi|_{R_2[x]})$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} z_1 = z_4 \\ z_2 = z_4 \\ z_3 = z_4 \end{array}$$

$$N(A) = \left\{ \begin{bmatrix} z_4 \\ z_4 \\ z_4 \\ z_4 \end{bmatrix}; z_4 \in \mathbb{R} \right\} = \mathcal{L} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

coefficients $\text{red } \{E_{11}, E_{12}, E_{21}, E_{22}\}$

PAZI, DA PREVEDEŠ MAZAJ!!!

7. predavanje

Monday, 20 November 2023 11:08

U vektorski prostor $\mathbb{R}_3[x]$ definiran $U = \{p \in \mathbb{R}_3[x]; p(-x) = -p(x)\}$
 a) Pokazi, da je U vektorski podprostor

$$p, q \in U \quad \alpha, \beta \in \mathbb{R}$$

$$(\alpha p + \beta q)(-x) = (\alpha p)(-x) + (\beta q)(-x) = \alpha p(-x) + \beta q(-x) = \alpha(-p(x)) + \beta(-q(x)) = -(\alpha p + \beta q)(x) \quad \checkmark$$

b) Izberi si bazu B prostora U .

$$p(x) = ax^3 + bx^2 + cx + d$$

$$\begin{aligned} p(-x) &= -p(x) \\ \cancel{a}(-x)^3 + \cancel{b}x^2 + \cancel{c}(-x) + d &= -(\cancel{a}x^3 + \cancel{b}x^2 + \cancel{c}x + d) \\ 2bx^2 + 2d &= 0 \\ bx^2 + d &= 0 \\ \swarrow \quad \searrow \\ b &= 0 \quad d = 0 \end{aligned}$$

$$U = \{p(x) = ax^3 + cx; a, c \in \mathbb{R}\} = \mathcal{L}\{x, x^3\}$$

↓ STA NEODVISNA
 $B_U = \{x, x^3\}$

c) Napiši $\phi: \mathbb{R}_3[x] \rightarrow \mathbb{R}_3[x]$. Ali je linearna?
 $\phi(p) = xp'' - 2p'$

$$\begin{aligned} \phi(\alpha p + \beta q) &= x(\alpha p + \beta q)'' - 2(\alpha p + \beta q)' \\ &= x(\alpha p'' + \beta q'') - 2(\alpha p' + \beta q') \\ &= \alpha(xp'' - 2p') + \beta(xq'' - 2q') \\ &= \alpha\phi(p) + \beta\phi(q) \quad \checkmark \end{aligned}$$

TUKI MAJMO
USE POLINOME

d) Zapiši matricu, ko pripada ϕ iz $\mathcal{B} = \{1, x, x^2, x^3\}_U$.

$$\phi(1) = x \cdot 1'' - 2 \cdot 1' = 0$$

$$\phi(x) = x \cdot x'' - 2 \cdot x' = -2$$

$$\phi(x^2) = x(x^2)'' - 2(x^2)' = 2x - 4x = -2x$$

$$\phi(x^3) = x(x^3)'' - 2(x^3)' = x(6x) - 2(3x^2) = 0$$

$$A_{\phi, \mathcal{B}, \mathcal{B}} = \begin{bmatrix} \phi(1) & \phi(x) & \phi(x^2) & \phi(x^3) \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix}$$

e) Izračunaj $\ker \phi$.

$$A = A_{\phi, \mathcal{B}, \mathcal{B}} = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N(A) = \{v \in \mathbb{R}^4: \hat{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, v_1 = v_2 = 0\}$$

$$A = A_{\phi, \gamma, \gamma} = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N(A) = \left\{ \vec{y} \in \mathbb{R}^4; \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, y_2 = y_3 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix}; y_1, y_4 \in \mathbb{R} \right\}$$

$$= \mathcal{L}\{\vec{e}_1, \vec{e}_4\}$$

$$\ker \phi = \{y_4 x^3 + y_3 x^2 + y_2 x + y_1; y_2 = y_3 = 0\}$$

$$= \{y_4 x^3 + y_1; y_1, y_4 \in \mathbb{R}\}$$

$$= \mathcal{L}\{1, x^3\}$$

f) $\gamma: U \rightarrow \mathbb{R}_3[x]$
 $\gamma(p) = x p'' - 2 p'$
 γ linearna? **DA, gledaj c)**

**TUKI MAMO SAMO
POLINOME IZ U**

g) Zapišemo matriko, ki pripada γ iz B v g.

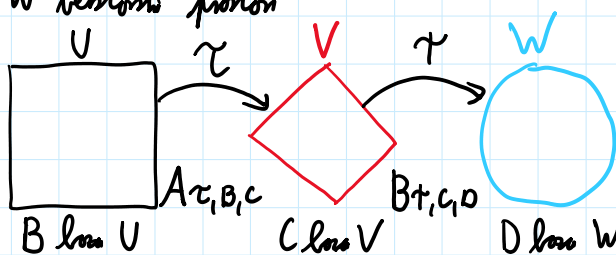
$$A_{\gamma, B, g} = \begin{matrix} & \phi(e_1) & \phi(e_3) \\ \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix} \end{matrix}$$

h) Določimo $\ker \gamma$.

$$\ker \gamma = \mathcal{L}\{x^3\}$$

KOMPOZITUMI LINEARNIH PRESLIKAV

U, V, W vektorski prostori



Naj bosta $\gamma: U \rightarrow V$ in $\tau: V \rightarrow W$ linearni. $(\tau \circ \gamma)(u) = \tau(\gamma(u))$
 $\tau \circ \gamma$ je linearna preslika $\tau \circ \gamma: U \rightarrow W$

$$(\tau \circ \gamma)(\alpha_1 u_1 + \alpha_2 u_2) = \tau(\gamma(\alpha_1 u_1 + \alpha_2 u_2))$$

$$= \tau(\alpha_1 \gamma(u_1) + \alpha_2 \gamma(u_2))$$

$$= \alpha_1 \tau(\gamma(u_1)) + \alpha_2 \tau(\gamma(u_2)) =$$

$$= \alpha_1 (\tau \circ \gamma)(u_1) + \alpha_2 (\tau \circ \gamma)(u_2) \quad \text{JE LINEARNA}$$

Če A matrika, ki pripada γ iz B v C
in B matrika, —||— γ iz C v D
matrika **RA** —||— $\gamma \circ \gamma$ iz B v D.

$\dim U = n$ $A \in \mathbb{R}^{n \times n}$
 $\dim V = m$ $B \in \mathbb{R}^{m \times n}$
 $\dim W = p$

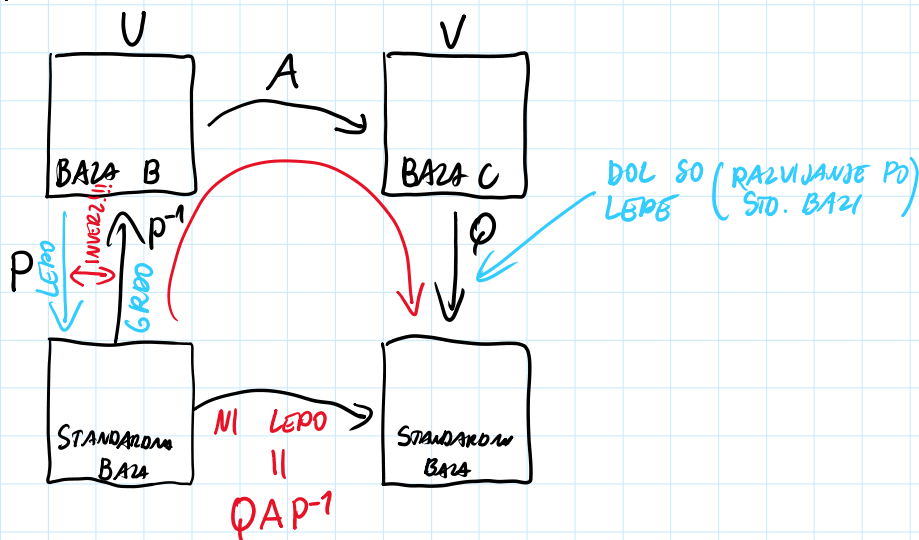
in B matrike, $-||-$ γ iz C v D $\dim V = n$ $B \in \mathbb{R}^{n \times n}$
 potem **BA** $-||-$ $\gamma \cdot \gamma$ iz B v D . $\dim W = p$

dokaz: $B = \{b_1, \dots, b_m\}$, $C = \{c_1, \dots, c_m\}$ in $D = \{d_1, \dots, d_p\}$. Izračunajmo $\gamma \circ \gamma$ iz B v D .

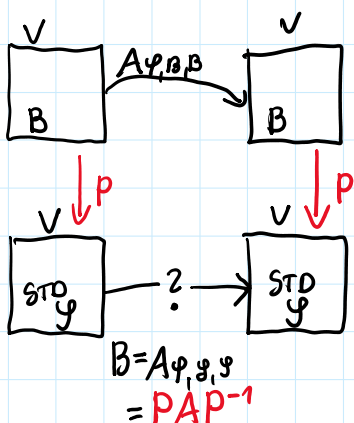
$$\begin{aligned}
 (\gamma \circ \gamma)(b_i) &= \gamma(\gamma(b_i)) = \gamma(a_{i1}c_1 + a_{i2}c_2 + \dots + a_{in}c_n) = \gamma\left(\sum_{i=1}^n a_{ii}c_i\right) \\
 &= \sum_{i=1}^n a_{ii} \gamma(c_i) = \sum_{i=1}^n a_{ii} \sum_{j=1}^p b_{ji}d_j = \\
 &= \sum_{j=1}^p \left(\sum_{i=1}^n a_{ii} b_{ji}\right) d_j \\
 &= \sum_{j=1}^p \left(\sum_{i=1}^n b_{ji} a_{ii}\right) d_j \\
 &= \sum_{j=1}^p (BA)_{ji} d_j
 \end{aligned}$$

Isto postopimo za stolpce 2, 3, 4, ..., m.

ZAKAJ?



$\varphi: V \rightarrow V$



Ure matrike, ki pripadajo $\varphi: V \rightarrow V$ iz neke baze B v B , so si med seboj **PODOBNE**.

PRIMER: Zapišimo matriko, ki ustrezno spodbija $\mathbb{Z}_n$ v $\mathbb{R}^3$ t.e. rešimo $x+2y+3z=0$ v
bazi po svoji želji.

$$Y = \{\vec{a}, \vec{b}, \vec{c}\}$$

$$\mathbb{Z}(\vec{a}), \mathbb{Z}(\vec{b}), \mathbb{Z}(\vec{c})?$$

TEŽKO!!

$$B = \{\vec{a}, \vec{b}, \vec{c}\}$$

↑
1. normirani
lin. nezávislá
↑
normála

$$\begin{aligned}\mathbb{Z}(\vec{a}) &= \vec{a} \\ \mathbb{Z}(\vec{b}) &= \vec{b} \\ \mathbb{Z}(\vec{c}) &= -\vec{c}\end{aligned}$$

$$A_{\mathbb{Z}, B, B} = \begin{bmatrix} \mathbb{Z}(\vec{a}) & \mathbb{Z}(\vec{b}) & \mathbb{Z}(\vec{c}) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix}$$

$$A \text{ ima lastni vektor } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \mathbb{Z} \text{ ima lastni vektor } \vec{c}$$

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$A_{\mathbb{Z}, Y, Y} = PAP^{-1}$$

LASTNE VREDNOSTI LINEARNIH PRESLIKAV

$$\tau: V \rightarrow V$$

$\vec{v}$ na $v \in V \neq 0$ in neko $\lambda \in \mathbb{C}$ velja $\tau(v) = \lambda v$ imenujemo
 λ lastna vrednost linearne preslikave in v lastni vektor.

Lastne vrednosti preslikave τ so natanko lastne vrednosti katerekoli
pripadajoče matrike.

$$\text{JEDRO } \text{KER } \tau \leftrightarrow N(A)$$

$$\text{SLIKA } \text{IM } \tau \leftrightarrow C(A)$$

$$\text{im}(\tau) = \{u \in U, \exists v \in V \Rightarrow \tau(v) = u\} \subseteq U$$

LASTNOSTI: 1) $\text{im}(\tau)$ je vektorski podprostor

$$\left. \begin{aligned} 2) u \in U \quad u = \beta_1 u_1 \dots \beta_m u_m \\ v \in V \quad v = d_1 v_1 \dots d_n v_n \end{aligned} \right\} A_{\tau, A, B} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

$$\text{im } \tau \leftrightarrow C(A)$$

i) Določimo $\text{im}(\Phi)$

$$\Phi(p) = xp'' - 2p'$$

$$C(A_{\Phi, Y, B}) = \mathcal{L} \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathcal{L} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$C(A_{\phi, \psi, \beta}) = \mathcal{L} \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\} = \mathcal{L} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\downarrow$$

$$\text{im}(\phi) = \mathcal{L}\{1, x\}$$

$$3) \text{im}(\tau) = U \Leftrightarrow \tau \text{ ist surjektiv}$$

$$4) \dim N(A) + \dim C(A) = \# \text{ Spalten } A$$

$$\downarrow$$

$$\dim \ker(\tau) + \dim \text{im}(\tau) = \dim V$$

$$5) \dim U = \dim V \Leftrightarrow \tau \text{ ist injektiv} \Leftrightarrow \ker \tau = \{0\}$$

$$\Leftrightarrow \dim \text{im}(\tau) = \dim V$$

$$\Leftrightarrow \dim \text{im}(\tau) = \dim U$$

$$\Leftrightarrow \text{im } \tau = U$$

$$\Leftrightarrow \tau \text{ surjektiv}$$

FUNKCIJE IN VEKTORSKE FUNKCIJE VEČ SPREMENLJIVK

$$f(x) = ax$$

$$f(x, y, z) = ax + by + cz = [a \ b \ c] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

POSLEJ: $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$\vec{F}(\underbrace{x_1, \dots, x_n}_{\text{SPREMENLJIVKE}}) = \vec{F}(\mathbf{x}) = \vec{F}(\underline{x}): \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

n	1	2	3	
1	$f: \mathbb{R} \rightarrow \mathbb{R}$	$f(x, y)$	$f(x, y, z)$	← FUNKCIJE VEČ SPREMENLJIVK $\mathbb{R}^n \rightarrow \mathbb{R}$
2	$f: \mathbb{R} \rightarrow \mathbb{R}^2$ $f(x) = \begin{bmatrix} x(x) \\ y(x) \end{bmatrix}$	$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$		
3	$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$			
	↑ PARAMETRIZIRANE KRIVULJE V n -DIMENZIJI			$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

POMOVI MO ZA $n=1$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$f(x_1, \dots, x_n)$$

$$f_{x_1 x_2 x_3 x_4 x_5} = \frac{\partial^5 f}{\partial x_1 \partial x_2 \partial x_3 \partial x_4 \partial x_5}$$

$$f(x, y, z) = x^2 + y^3 + e^{x \cos(z)}$$

$$\left. \begin{aligned} \frac{\partial f}{\partial x} &= f_x = 2x + 0 + \cos(z) e^{x \cos(z)} \\ \frac{\partial f}{\partial y} &= f_y = 3y^2 \\ \frac{\partial f}{\partial z} &= f_z = -e^{x \cos(z)} x \sin(z) \end{aligned} \right\} \text{PARCIALNI ODVODI}$$

$$f_{xy} = f_{yx}, \text{ to } f_{xy} \text{ mena } \text{Če } f_{x_i x_j} \text{ mena } \Rightarrow f_{x_i x_j} = f_{x_j x_i}$$

$$(\text{grad } f)(\underline{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

↑ dari točki x_0 je **največji preostek funkcije** f v smeri $(\text{grad } f)(x_0)^T$.

HESSEJEVA MATRIKA

HESSEJEVA MATRIKA

$$H_f(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right]_{i,j=1 \dots n}$$

Če ni $\frac{\partial^2 f}{\partial x_i \partial x_j}$ neni, potem je $H_f(x)$ simetrična.

NOVO $n \geq 2$

$$\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\vec{F}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \text{ kjer je } f_i: \mathbb{R}^n \rightarrow \mathbb{R}$$

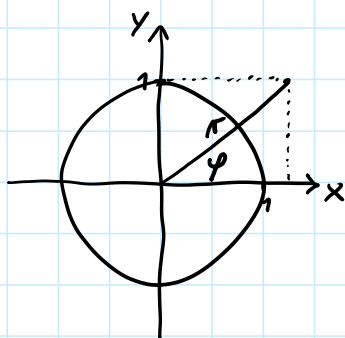
JACOBIJEVA MATRIKA prihodstvenih vektorskih funkcij

$$J_F = \left[\frac{\partial f_i}{\partial x_j}(x) \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

$\downarrow$
 SPREMEMBA $\vec{F}$ PRI
 MALI SPREMEMBI x_1

$\leftarrow \text{grad } f_1$
 $\leftarrow \text{grad } f_m$

PRIMER: cilindrične koordinate v $\mathbb{R}^3$

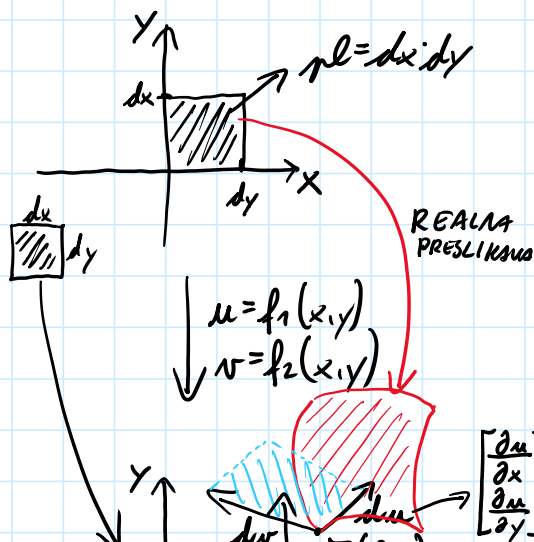


$$\left. \begin{aligned} x &= r \cos(\varphi) \\ y &= r \sin(\varphi) \\ z &= z \end{aligned} \right\} \text{POLARNE KOORDINATE}$$

Jacobijska matrika polarnih koordinat

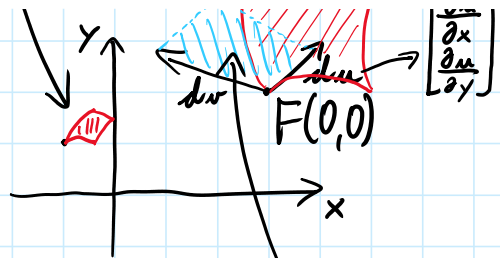
$$\vec{F}(r, \varphi, z) = \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \\ z \end{bmatrix} \leftarrow \begin{aligned} f_1(r, \varphi, z) \\ f_2(r, \varphi, z) \\ f_3(r, \varphi, z) \end{aligned}$$

$$J_{\text{VALNE}}(r, \varphi, z) = \begin{bmatrix} \cos \varphi & -r \sin \varphi & 0 \\ \sin \varphi & r \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{aligned} x \\ y \\ z \end{aligned}$$



$$J_{\text{VALNO}}(r, \varphi, z) = \begin{vmatrix} \cos \varphi & -r \sin \varphi & 0 \\ \sin \varphi & r \cos \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{matrix}$$

$\uparrow$ r $\uparrow$ φ $\uparrow$ z



Absolutna vrednost determinante Jacobije nam pove faktor prostornosti **LOKALNO** po in pred preslikavo.

NAJBOLJŠA
LINEARNA PRESLIKAVA

$$pl = |\det J_{u,v}| dx dy$$

$$\det J_{\text{VALNO}}(r, \varphi, z) = r \cos^2 \varphi + r \sin^2 \varphi = r$$

$$\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\vec{F}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

$$J_{\vec{F}}(x) = \left[\frac{\partial f_i}{\partial x_j}(x) \right]_{\substack{i=1 \dots m \\ j=1 \dots n}} = \boxed{\frac{\partial \vec{F}}{\partial x}}$$

ODVOD FUNKCIJE
PO VEKTORJU

Primeri funkcij:

1) linearna:

$$\vec{F}(x) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Ax^T$$

Kaj je $\frac{\partial \vec{F}}{\partial x}$?

$$\left(\frac{\partial \vec{F}}{\partial x} \right)_{i,j} = \frac{\partial f_i}{\partial x_j} = \frac{\partial (a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n)}{\partial x_j} = a_{ij}$$

$$\boxed{\frac{\partial (Ax)}{\partial x} = A}$$

$$m=1: \frac{\partial (x^T \vec{a})}{\partial x} = \vec{a}^T \quad m=n=1: \frac{\partial (ax)}{\partial x} = a$$

2) $\vec{F}(\vec{x}) = \vec{x}^T A \vec{x}$ za neko $A \in \mathbb{R}^{n \times n}$

KVAADRATNA
FORMA

$$\vec{F}(\vec{x}) \stackrel{m=1}{=} \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_i x_j \quad \frac{\partial \vec{F}}{\partial x_k} = \text{grad}(\vec{x}^T A \vec{x})_k = \frac{\partial (\vec{x}^T A \vec{x})}{\partial x_k}$$

$$\begin{aligned} \frac{\partial \vec{F}}{\partial x_k} &= \frac{\partial (x_k^2 + x_k x_1 + x_k x_2 + \dots + x_k x_n)}{\partial x_k} \\ &= 2a_{kk}x_k + (a_{1k} + a_{k1})x_1 + (a_{2k} + a_{k2})x_2 + \dots + (a_{nk} + a_{kn})x_n \\ &= \sum_{i=1}^n (a_{ik} + a_{ki})x_i \\ &\hat{=} \quad \quad \quad (A^T + A) \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$

$$= \sum_{i=1}^n (a_{ik}^T a_{ki}) x_i$$

$$= \sum_{i=1}^n a_{ik} x_i + \sum_{i=1}^n a_{ki} x_i = (A^T \vec{x} + A \vec{x})_k = ((A^T + A) \vec{x})_k$$

$$\begin{array}{cc} \square & \equiv \\ (A^T \vec{x})_k & (A \vec{x})_k \end{array}$$

$$\frac{\partial (\vec{x}^T A \vec{x})}{\partial \vec{x}} = ((A + A^T) \vec{x})^T \xrightarrow{\text{SIMMETRIE}} 2 \vec{x}^T A$$

$$3) A = I$$

$$\frac{\partial \|\vec{x}\|^2}{\partial \vec{x}} = \frac{\partial (\vec{x}^T \vec{x})}{\partial \vec{x}} = 2 \vec{x}^T$$

4) KOMPOZITUM

$$\mathbb{R}^n \xrightarrow[\frac{\partial \vec{F}}{\partial \vec{x}}]{\vec{F}} \mathbb{R}^m \xrightarrow[\frac{\partial \vec{G}}{\partial (\vec{F}(\vec{x}))}]{\vec{G}} \mathbb{R}^p$$

$$\searrow \vec{G} \circ \vec{F} \rightarrow$$

$$\frac{\partial (\vec{G} \circ \vec{F})}{\partial \vec{x}} = \frac{\partial \vec{G}}{\partial \vec{y}} \frac{\partial \vec{F}}{\partial \vec{x}}$$

$$f(x) = \|5\vec{x}\|^2$$

$$\begin{array}{l} \vec{x} \rightarrow 5\vec{x} \rightarrow \|5\vec{x}\|^2 \\ \mathbb{R}^n \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^1 \end{array}$$

$$\frac{\partial f}{\partial \vec{x}} = 2(5\vec{x})^T \cdot 5I = \underline{\underline{50\vec{x}^T}}$$

VECKRATNI INTEGRALI

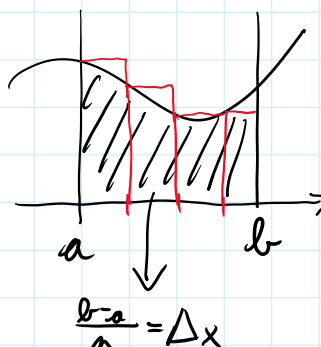
$$f: \mathbb{R} \rightarrow \mathbb{R}$$

NEDOLOČEN INTEGRAL $f: F: \mathbb{R} \rightarrow \mathbb{R}$
 $F'(x) = f$

DOLOČENI INTEGRAL $f: \int_a^b f(x) dx$
 $= F(b) - F(a)$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} f(x_i^*)$$

NAKLJUČNA TOČKA
NA INTERVALU (x_i, x_{i+1})



TEGA NE BO ZAHTEVALA:

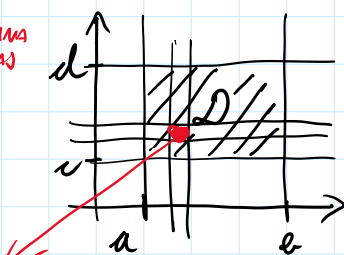
$$f: \mathbb{R}^2 \rightarrow \mathbb{R} \quad D \in \mathbb{R}^2 \quad D = [a, b] \times [c, d]$$

DOLOČENI INTEGRAL, PREDZNAČENA PROSTORNA
 POD $z = f(x, y)$ ZNOTRAJ

$$\iint_D f(x, y) dx dy =$$

$$= \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \frac{b-a}{n} \frac{d-c}{m} f(x_i^*, y_j^*)$$

$$(x_i^*, y_j^*) \in D_{ij}$$

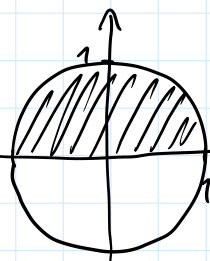


Ali jo pretvoriti **DVOJNI** integral na **DVAKRATNI** integral?

$$\iint_{D \in \mathbb{R}^2} f(x, y) dx dy \quad \downarrow \quad \int_{D_y \in \mathbb{R}} \left(\underbrace{\int_{D_x \in \mathbb{R}} f(x, y) dx}_{\text{FUNKCIJA SPREMENLJIVE } x} \right) dy$$

PRIMERI:

$$\iint_{\substack{\text{ZG. POLOVICA} \\ \text{ENOTSKEGA KROGA}}} \sqrt{1-x^2-y^2} dx dy = \frac{1}{4} \text{ PROSTORNIŠE} \\ \text{ENOTSKE KROGE} \\ = \frac{1}{4} \cdot \frac{4}{3} \pi \cdot 1^3 \\ = \frac{\pi}{3}$$



$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$x^2+y^2+z^2 = 1 - \text{ENOTSKA SFERA}$$

FUBINIJEV IZREK:

① $D \in [a, b] \times [c, d]$ prostotnih

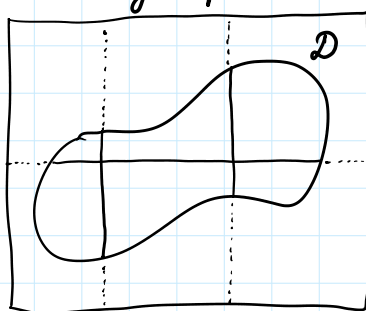
$$\underbrace{\iint_D f(x, y) dx dy}_{\text{DVOJNI INTEGRAL}} = \int_{c, d} \left(\int_{a, b} f(x, y) dx \right) dy$$

$$\underbrace{\iint_D f(x,y) dx dy}_{\text{DVOJNI INTEGRAL PO PRAVOKOTNIKU}} = \int_a^b \left(\int_a^d f(x,y) dx \right) dy$$

Integriraj $f(x,y) = xe^y$ nad $D = [3,4] \times [0,1]$

$$\begin{aligned} \iint_{[3,4] \times [0,1]} xe^y dx dy &= \int_{[0,1]} \left(\int_{[3,4]} xe^y dx \right) dy = \int_{[0,1]} \left(\int_{[3,4]} ex^y dy \right) dx \\ &= \int_0^1 \left(e^{\frac{y}{2}} \frac{x^2}{2} \Big|_{x=3}^4 \right) dy = \int_0^1 \left(8 - \frac{9}{2} \right) e^{\frac{y}{2}} dy \\ &= \frac{7}{2} \int_0^1 e^{\frac{y}{2}} dy = \frac{7}{2} [e^{\frac{y}{2}}]_0^1 = \frac{7}{2} (e - 1) \end{aligned}$$

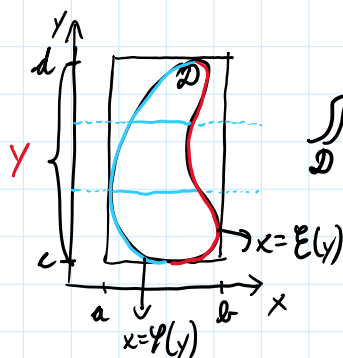
② D ni nujno pravokotnik



P pravokotnik

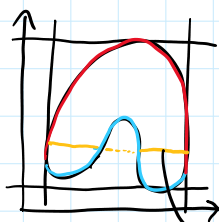
$$\iint_D f(x,y) dx dy = \iint_P g(x,y) dx dy$$

$$\text{tjer } g(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \in P \setminus D \end{cases}$$



$$\iint_D f(x,y) dx dy = \int_c^d \left(\int_a^{E(y)} f(x,y) dx \right) dy$$

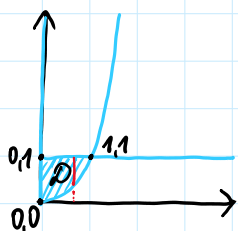
FIKSA STAVILA $\uparrow$ FUNKCIJA $\uparrow$



$$\iint_D f(x,y) dx dy = \int_a^b \left(\int_{P(x)}^{E(x)} f(x,y) dy \right) dx$$

Če li obliki nista gredo...

Imajmo $\iint_D \frac{x}{y} dx dy$ po območju D , omejeno z grafki funkcij $y=x^2$, $y=1$, $x \geq 0$



a) ploščina?

$$\int_0^1 1 dx - \int_0^1 x^2 dx \quad (\text{ZGORNJA - SPODNJA})$$

$$\iint_D dx dy = \int_0^1 \left(\int_{x^2}^1 dy \right) dx = \int_0^1 (y|_{x^2}^1) dx = \int_0^1 (1 - x^2) dx = \int_0^1 1 dx - \int_0^1 x^2 dx$$

$$\iint_D dx dy = \int_0^1 \left(\int_0^{1-x^2} dy \right) dx = \int_0^1 (y|_{x^2}^0) dx = \int_0^1 (1-x^2) dx = \int_0^1 1 dx - \int_0^1 x^2 dx$$

$$= x \Big|_0^1 - \frac{x^3}{3} \Big|_0^1 = 1 - \frac{1}{3} = \underline{\underline{\frac{2}{3}}}$$

$$\iint_D \frac{x}{y} dx dy = \int_0^1 \left(\int_0^{1-y} \frac{x}{y} dx \right) dy = \int_0^1 \left(\frac{1}{y} \frac{x^2}{2} \Big|_0^{1-y} \right) dy = \int_0^1 \left(\frac{1}{y} \cdot \frac{1}{2} \right) dy = \frac{1}{2} \int_0^1 \frac{1}{y} dy = \frac{1}{2} \Big|_0^1 = \underline{\underline{\frac{1}{2}}}$$

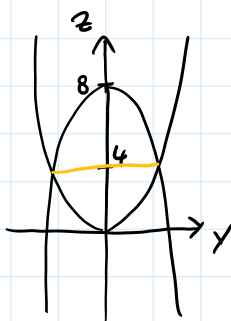


$$= \int_0^1 \left(\int_{x^2}^1 \frac{x}{y} dy \right) dx = \int_0^1 x \left[\log|y| \right]_{x^2}^1 dx = \int_0^1 x (0 - \log(x^2)) dx = 2 \int_0^1 x \log x dx =$$

PER PARTES (GA JE TREBA)
+ L'HOSPITAL ZNAT

PRVI KORAK VEDNO NARISI OBRMOČJE!!!

Dana su paraboloida $z = x^2 + y^2$ in $z = 8 - x^2 - y^2$. Uložiš med njima omejeno $\mathcal{D}$.
Iščunajto prostorno območje $\mathcal{D}$.



1. NARISI
2. IZRAČUNAJ PRESEČIŠČE

$$z = x^2 + y^2$$

$$z = 8 - x^2 - y^2$$

$$2z = 8$$

$$z = 4 \Rightarrow x^2 + y^2 = 4$$

KROŽNICA RADIJ 2

③ NAPISI INTEGRAL in IZRAČUNAJ

$$\iiint_D dx dy dz = \int_{-2}^2 \left(\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(\int_{x^2+y^2}^{8-x^2-y^2} dz \right) dx \right) dy$$

$\mathcal{D}: x^2 + y^2 \leq z \leq 8 - x^2 - y^2$

$x^2 + y^2 \leq 4$ GRAD

↓ VALJNO KOORDINATE

$$x^2 + y^2 = r^2 \Rightarrow \begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$

$$z = z$$

$$\iiint_D dx dy dz = \iiint_D \underbrace{r}_{|\det J|} dr d\varphi dz = \int_0^2 \left(\int_0^{2\pi} \left(\int_{r^2}^{8-r^2} r dz \right) d\varphi \right) dr$$

$$= \int_0^2 \left(\int_0^{2\pi} \left(r z \Big|_{z=r^2}^{8-r^2} \right) d\varphi \right) dr$$

$$= \int_0^2 \left(\int_0^{2\pi} r (8 - r^2 - r^2) d\varphi \right) dr$$

$$= \int_0^2 \left(\int_0^{2\pi} (8r - 2r^3) d\varphi \right) dr$$

$$= \int_0^2 (8r - 2r^3) \varphi \Big|_{\varphi=0}^{2\pi} dr$$

$$r^2 \leq z \leq 8 - r^2$$

$$r^2 \leq 4$$

$$\begin{aligned}
&= \int_0^2 (8r - 2r^3) \varphi \Big|_{\varphi=0}^{2\pi} dr \\
&= 2\pi \int_0^2 (8r - 2r^3) dr \\
&= 2\pi \left(4r^2 - \frac{r^4}{2} \right) \Big|_{r=0}^{r=2} \\
&= 2\pi (16 - 8) = \underline{\underline{16\pi}} \\
&\underline{\underline{V = 16\pi}}
\end{aligned}$$

LOKALNI EKSTREMI

ponovimo:
 $f: \mathbb{R}^1 \rightarrow \mathbb{R}$

NI EKSTREM
 (PREVOJ/SEDOLO)

Če f odredljiva $\rightarrow f'(x) = 0$

OBRAZ NE VELJA!!!

Kdaj je stacionarna točka lok. MIN, lok. MAX ali nič od tega?

$f''(x) > 0 \rightarrow f$ v x je KONVEKSIJNA.
 $f''(x) < 0 \rightarrow f$ v x je KONKAVNA.

- $f''(a) < 0$, f ma **LOKALNI MAKSYMUM**
- $f''(a) > 0$, f ma **LOKALNI MINIMUM**
- $f''(a) = 0$, f ma **KRZKOLCI** (głównie noszący nazwę $f''(a)$, $f'''(a)$...)

$$f(a+h) \approx f(a) + h \cdot f'(a) + \frac{1}{2} h^2 f''(a) + \dots + \frac{1}{k!} h^k f^{(k)}(a) \quad \text{TAYLORJEVA VRSTA}$$
$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$f(\underline{a} + \underline{h}) = f(a_1 + h_1, a_2 + h_2) \approx f(\underline{a}) + h_1 \frac{\partial f}{\partial x_1}(\underline{a}) + h_2 \frac{\partial f}{\partial x_2}(\underline{a}) + \frac{1}{2} (h_1^2 \frac{\partial^2 f}{\partial x_1^2} + h_2^2 \frac{\partial^2 f}{\partial x_2^2} + h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + h_1 h_2 \frac{\partial^2 f}{\partial x_2 \partial x_1}) + \dots$$
$$(\text{grad } f)(a) \cdot \underline{h}^T \quad \underline{h}^T H_f(a) \underline{h}^T \quad \text{KVAADRATNA FORMA}$$

Naj bo $f: \mathbb{R}^n \rightarrow \mathbb{R}$ dvakrat **ZVEZNO** odvedljiva, a stacionarna točka:

$-f(a+h) > 0$ sa nae h - **MINIMUM**
 $-f(a+h) < 0$ sa nae h - **MAKSIMUM**

$$f(a+h) \approx f(a) + \underbrace{(\text{grad } f)(a)}_0 h^T + \frac{1}{2} h H_f h^T$$

$\rightarrow H_f(a)$ PD $\Leftrightarrow b H_f b^T > 0$ na na h
 $\rightarrow H_f(a)$ ND $\Leftrightarrow b H_f b^T < 0$ na na h

$f: \mathbb{R}^n \rightarrow \mathbb{R}$, naj. $2x$ sono svedljiva in a njea stacionarna točka:

a) $H_f(u)$ PD $\Leftrightarrow a$ je lokalni MINIMUM

b) $\text{H}_2(\text{u})$ NO $\Leftrightarrow$ a je lokahi: MAKSIMUM

c) $\text{He}(\text{s})$ redefinitiv $\Leftrightarrow$ a ni lokalni ekstrem (SEOLO)

noji. Za svako a zadovoljna je a njena stacionarna tačka:

- a) $H_f(a)$ PD $\Leftrightarrow a$ je lokalni MINIMUM
- b) $H_f(a)$ NO $\Leftrightarrow a$ je lokalni MAKSIUM
- c) $H_f(a)$ indefinita $\Leftrightarrow a$ ni lokalni ekstrem (SEOLO)
- d) C je set $\{H_f(a) = 0\}$, gde C je prazna ili drugi specijalni slučaj nivoa deložij informacij

način je najbrži, ali mora se razmišljati o drugih pomembnih informacijah

Naj bo $b \in \mathbb{R}$ in $f(x,y) = 2x^2 + y^2 + bxy$. Identificiraj vse lokalne ekstreme.

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x + by \\ \frac{\partial f}{\partial y} &= 2y + bx \end{aligned} \quad \begin{aligned} 4x + by &= 0 \rightarrow x = -\frac{by}{4} \\ 2y + bx &= 0 \\ 2y + b\left(-\frac{by}{4}\right) &= 0 \\ 2y - \frac{b^2y}{4} &= 0 \\ y\left(2 - \frac{b^2}{4}\right) &= 0 \\ y=0, x=0 &\rightarrow T_1(0,0) \\ 2 - \frac{b^2}{4} &= 0 \rightarrow b = \pm 2\sqrt{2} \end{aligned} \quad \begin{aligned} 4x \pm 2\sqrt{2}y &= 0 \\ 2y \pm 2\sqrt{2}x &= 0 \rightarrow \pm 2\sqrt{2} \\ 2y &= \pm 2\sqrt{2}x \\ y &= \pm x\sqrt{2} \end{aligned}$$

$$T_2(x, \pm x\sqrt{2}), \text{ za } b = \pm 2\sqrt{2}$$

$$\left. \begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 4 \\ \frac{\partial^2 f}{\partial x \partial y} &= b \\ \frac{\partial^2 f}{\partial y \partial x} &= b \\ \frac{\partial^2 f}{\partial y^2} &= 2 \end{aligned} \right\} H_f(x,y) = \begin{bmatrix} 4 & b \\ b & 2 \end{bmatrix} \text{ NEODVISNA OD } x \text{ in } y$$

$$H_f(T_1) = \begin{bmatrix} 4 & b \\ b & 2 \end{bmatrix} \quad \begin{aligned} \det H_f &= 8 - b^2 > 0 \Rightarrow \text{PD} \\ b^2 &< 8 \\ -2\sqrt{2} &< b < 2\sqrt{2} \end{aligned}$$

Če $b \in (-2\sqrt{2}, 2\sqrt{2})$, T_1 je lokalni **MINIMUM**.

Če $b > 2\sqrt{2}$ in $b < -2\sqrt{2}$, T_1 ni lokalni ekstrem (**SEDLO**)

$$b = \pm 2\sqrt{2} \quad H_f(T_2) = \begin{bmatrix} 4 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & 2 \end{bmatrix} \rightarrow \det = 0 \rightarrow \text{NE VEMO NIČESAR}$$

MALO DRUGAČE:

$$f(x,y) = 2x^2 + y^2 + bxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{b}{2} \\ \frac{b}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

opomba: $f: \mathbb{R}^n \rightarrow \mathbb{R}$ $f(x) = x^T M x^T$
Preverimo, da $\text{grad } f(a) = 0$

$$f(x) = x^T M x \quad J_f = 2x^T M \rightarrow H_f = J_{\text{grad } f}$$

$$\begin{aligned} H_f &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)^T = \frac{\partial}{\partial x} (2x^T M)^T \\ &= \frac{\partial}{\partial x} (2M^T x) \\ &= 2 \frac{\partial}{\partial x} M x \\ &= 2M \rightarrow \begin{bmatrix} 4 & b \\ b & 2 \end{bmatrix} \end{aligned}$$

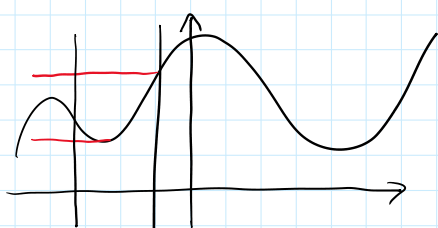


$H_f(x)$ je PD



$H_f(x)$ je ND

EKSTREMI PRI DODATNIH POGOJIH



MAKS FUNKCIJE f
NA INTERVALU (a, b)

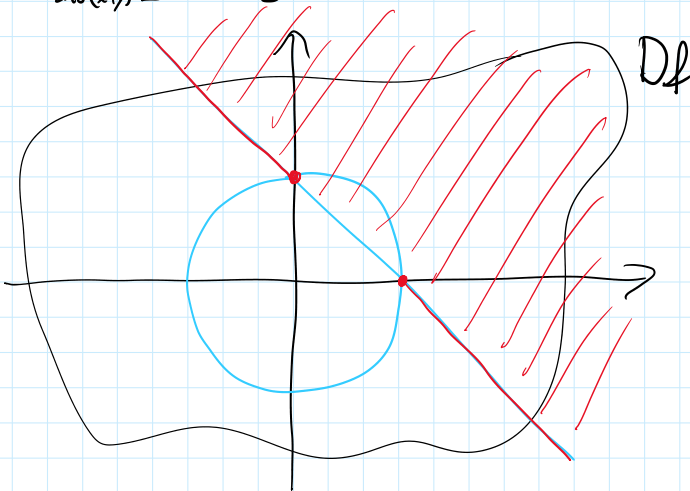


$f: \mathbb{R}^n \rightarrow \mathbb{R} \leftarrow$ ŽELIMO OPTIMIZIRATI

$g_i, h_j: \mathbb{R}^n \rightarrow \mathbb{R} \leftarrow$ ROBNJI POGOJI

Isčemo min/max $f(x)$, pri pogojih $g_i(x) = 0$ za $i = 1 \dots m$
 $h_j(x) \leq 0$ za $j = 1 \dots p$

$n=2$ } $f(x,y) \leftarrow$ MAX/MIN VREDNOSTI
 $m=2$ } $g_1(x,y) = 0$ $x+y=2$
 $n=3$ } $g_2(x,y) = 0$ $x^2+y^2=4$ } POGOJI, KI ZMANJSAJO
 $h_1(x,y) \leq 0$ $x+y \geq 2$ } DEFINICIJSKO OBMOČJE
 $h_2(x,y) \leq 0$ $\vdots$
 $h_3(x,y) \leq 0$ $\vdots$



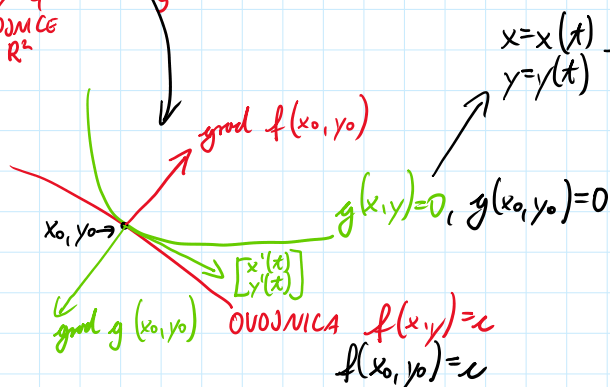
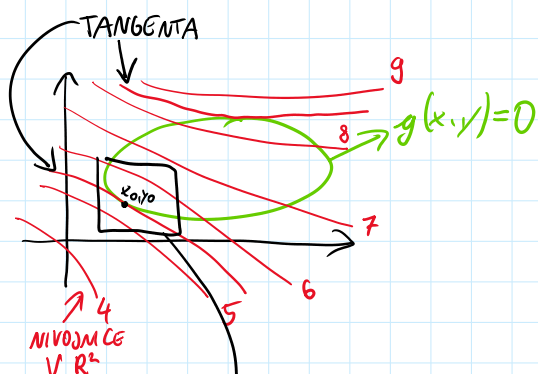
OPTIMIZACIJA Z ROBNIMI POGOJI OBLIKE $\vec{G}(x)=0$

PROBLEM: $f: \mathbb{R}^n \rightarrow \mathbb{R} \leftarrow$ realna optimizacija
 $g_i: \mathbb{R}^n \rightarrow \mathbb{R} \leftarrow$ robni pogoji

Površina najmanjše/največje vrednote pri pogojih:

Čaj se zgodijo, kar $n=2$ in $m=1$

max/min $f(x,y)$ pri pogoju $g(x,y)=0$ (obse namo sledilnice)



$$x=x(t) \rightarrow \text{odvodi po } t \rightarrow \frac{dg}{dx} \frac{dx}{dt} + \frac{dg}{dy} \frac{dy}{dt} = 0$$

$$(\text{grad } g)(x_0, y_0) \cdot \begin{bmatrix} x'(t_0) \\ y'(t_0) \end{bmatrix} = 0$$

V točki (x_0, y_0) , kjer f doseže MAX/MIN pri dodatni pogoj $g(x,y)=0$ velja, da sta

$$(\text{grad } f)(x_0, y_0) \parallel (\text{grad } g)(x_0, y_0)$$

VZPOREDNA

POGOJ: $(\text{grad } f)(x_0, y_0) \neq 0$ in $(\text{grad } g)(x_0, y_0) \neq 0$

$$(\text{grad } f)(x_0, y_0) = \lambda (\text{grad } g)(x_0, y_0) *$$

Definiramo $L(x, y, \lambda) = f(x, y) - \lambda g(x, y) \leftarrow$ LAGRANGEVA FUNKCIJA

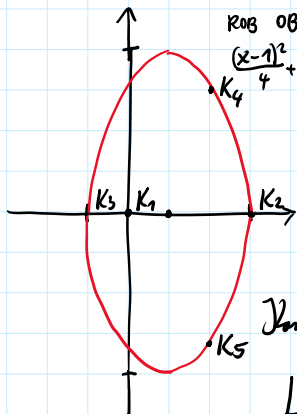
LAGRANGER MULTIPLIKATOR

$$\begin{array}{l} \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \\ \frac{\partial L}{\partial \lambda} = -g(x, y) \end{array} \xrightarrow{\text{VREDNOSTI V } x_0, y_0} \left\{ \begin{array}{l} (\text{grad } f)(x_0, y_0) - \lambda (\text{grad } g)(x_0, y_0) \stackrel{*}{=} 0 \\ 0 \end{array} \right.$$

V točki (x_0, y_0) , kjer f doseže MAX/MIN pri dodatni pogoj $g(x,y)=0$ in

V točki (x_0, y_0) , kjer f doseže MAX/MIN pri določenem pogoju $g(x, y) = 0$ ima
 $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ stacionarno točko.

V katerih točkah na območju $4(x-1)^2 + y^2 \leq 16$ zares funkcionira
 $f(x, y) = 2x^2 + y^2$ najmanjšo/največjo vrednost?



ROB OBMOČJA
 $\frac{(x-1)^2}{4} + \frac{y^2}{16} = 1$

Ali so kandidati v območju?

$$(\text{grad } f)(x, y) = (4x, 2y)$$

$$(\text{grad } f)(x, y) = 0$$

$$(x, y) = (0, 0) \quad K_1(0, 0) \rightarrow 4(0-1)^2 + 0^2 < 16 \quad \checkmark$$

LEŽI V NOTRAJNOSTI

Kandidati na robu?

$$L(x, y, \lambda) = 2x^2 + y^2 - \lambda(4(x-1)^2 + y^2 - 16)$$

$$\frac{\partial L}{\partial x} = 4x - 4\lambda \cdot 2(x-1)$$

$$\frac{\partial L}{\partial y} = 2y - 2\lambda y$$

$$\frac{\partial L}{\partial \lambda} = -(4(x-1)^2 + y^2 - 16)$$

$$4x - 8\lambda x + 8\lambda = 0 \rightarrow x - 2\lambda x + 2\lambda = 0$$

$$2y - 2\lambda y = 0 \rightarrow y - \lambda y = 0$$

$$4(x-1)^2 + y^2 - 16 = 0$$

$$y(1-\lambda) = 0$$

$$y = 0 \quad \lambda = 1$$

$$a) y = 0$$

$$4(x-1)^2 = 16$$

$$(x-1)^2 = 4$$

$$x-1 = \pm 2$$

$$x = 3 \quad x = -1$$

PREVERI
LAMBDO

$$3 - 6\lambda + 2\lambda = 0$$

$$4\lambda = 3$$

$$\lambda = \frac{3}{4}$$

$$-1 + 2\lambda + 2\lambda = 0$$

$$\lambda = \frac{1}{2}$$

$$K_2(3, 0) \quad K_3(-1, 0)$$

LAMBDA = 1

$$b) \lambda = 1$$

$$4x - 8x + 8 = 0$$

$$4x = 8$$

$$x = 2$$

$$K_4(2, 2\sqrt{3})$$

$$4(2-1)^2 + y^2 = 16$$

$$y^2 = 12$$

$$y = \pm 2\sqrt{3}$$

$$K_5(2, -2\sqrt{3})$$

$$f(K_1) = 0 \leftarrow \text{NAJNIZJA}$$

$$f(K_2) = 18$$

$$f(K_3) = 2$$

$$f(K_4) = f(K_5) = 20 \leftarrow \text{NAJVISJA}$$

V (0,0) najmanjšo, v $(2, \pm 2\sqrt{3})$ pa največjo.

SPLOŠNO:

$$\left. \begin{array}{l} \text{min/max} \\ \text{pri pogojih} \\ f(x) \\ g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_n(x) = 0 \end{array} \right\}$$

$$L(x, \lambda_1, \lambda_2, \dots, \lambda_n) = f(x) - \lambda_1 g_1(x) - \lambda_2 g_2(x) - \dots - \lambda_n g_n(x)$$

$$\vec{G}(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix} = 0$$

$$L(x, \lambda) = f(x) - \lambda \cdot \vec{G}(x)$$

MIN/MAX funkcij f pri robnih pogojih $\vec{G}(x) = \vec{0}$ je stacionarna točka

MIN/MAX funkcij f pri nekaterih pogojih $\vec{G}(\vec{x}) = \vec{0}$ je stacionarna točka

$$L(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \vec{\lambda}^T g(\vec{x})$$

$$(\text{grad } L)(\vec{x}^*, \vec{\lambda}^*) = 0$$

↑
RESEVANJE SISTEMA
M+N ENAČB z M+N
NEZNANKAMI

POGOTOJ: $(\text{grad } g_i)(\vec{x}^*) \neq 0$
 $\{(\text{grad } g_i)(\vec{x}^*)\}_{i=1 \dots m}$ so linearno neodvisni

PRIMER: $A \in \mathbb{R}^{m \times n}$, $m \leq n$ in $\text{rang}(A) = m$
 $\vec{b} \in \mathbb{R}^m$

Med vsemi rešenji $A\vec{x} = \vec{b}$ poišči tistega z najmanjšo normo?

ROBNI POGOJ: $A\vec{x} = \vec{b}$ (n pogojev) $\vec{G}(\vec{x}) = A\vec{x} - \vec{b} = \vec{0}$
 MAX/MIN : $f(\vec{x}) = \|\vec{x}\|^2$

$$L(\vec{x}, \vec{\lambda}) = \|\vec{x}\|^2 - \vec{\lambda}^T (A\vec{x} - \vec{b})$$

$\downarrow$ $\downarrow$
 $\mathbb{R}^n$ $\mathbb{R}^m$

$$\frac{\partial L}{\partial \vec{x}} = 2\vec{x}^T - \vec{\lambda}^T A = 0$$

$$\frac{\partial L}{\partial \vec{\lambda}} = -(A\vec{x} - \vec{b})^T = 0$$

$$2\vec{x}^T = \vec{\lambda}^T A$$

$$2\vec{x} = A^T \vec{\lambda}$$

$$A^T \vec{\lambda} = 2\vec{x}$$

$$\vec{x} = \frac{A^T \vec{\lambda}}{2}$$

M. MUJO DA
JE A OBRNJAVA

$$\vec{x} = A^{-1} \vec{b}$$

$$AA^T \vec{\lambda} = 2\vec{b}$$

$$\vec{\lambda} = 2(AA^T)^{-1} \vec{b}$$

$$\vec{x} = \frac{A^T}{2} \vec{\lambda}$$

$$\vec{x} = \frac{1}{2} A^T 2(AA^T)^{-1} \vec{b}$$

$$= A^T (AA^T)^{-1} \vec{b}$$

$$\frac{\partial (A\vec{x})}{\partial \vec{x}} = A$$

$$\frac{\partial (\vec{x}^T A \vec{x})}{\partial \vec{x}} = \vec{x}^T (A + A^T)$$

$$\frac{\partial \|\vec{x}\|^2}{\partial \vec{x}} = 2\vec{x}^T$$

OPTIMIZACIJA PRI ROBNIH POGOJIH (enakosti ali neenakosti)

min funkcijo $f(x)$
 pri pogojih $g_i(x) = 0$ za $i=1 \dots m$ $L(x, \underline{\lambda}) = f(x) - \underline{\lambda} \vec{G}(x)$
 $h_j(x) \leq 0$ za $j=1 \dots p$

$$D = D_f \cap \{x; g_i(x) = 0 \text{ za } i=1 \dots m\} \cap \{x; h_j(x) \leq 0 \text{ za } j=1 \dots p\}$$

Problem: min funkcijo f na območju D

Definiramo:

$$\vec{G}(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix} \quad \vec{H}(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}$$

$$\vec{G}: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\vec{H}: \mathbb{R}^n \rightarrow \mathbb{R}^p$$

Iščemo min $f(x)$ pri pogojih $\vec{G}(x) = \vec{0}$ $\vec{H}(x) \leq \vec{0}$ ^{po komponentah}

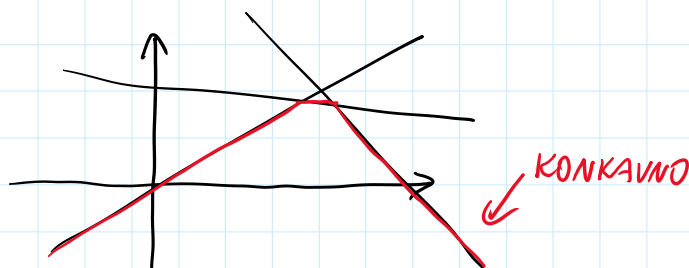
$$L(x, \underline{\lambda}, \underline{\mu}) = f(x) - \underline{\lambda} \vec{G}(x) - \underline{\mu} \vec{H}(x) \quad \mathbb{R}^{m+p} \rightarrow \mathbb{R}$$

$\underline{\lambda}, \underline{\mu}$ PRIREJENE
SPREMENLJIVKE

$$K(\underline{\lambda}, \underline{\mu}) = \min_x L(x, \underline{\lambda}, \underline{\mu}) = \min_x \{f(x) - \lambda_1 g_1(x) - \dots - \lambda_m g_m(x) - \mu_1 h_1(x) - \dots - \mu_p h_p(x)\}$$

LINEARNO V $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_p$

PRIREJENA
FUNKCIJA



Če iščemo x^* isto točko v D ($\vec{G}(x^*) = \vec{0}$ in $\vec{H}(x^*) \leq \vec{0}$), za katero je
 $\min_{x \in D} f(x) = f(x^*) = p^*$

$$K(\underline{\lambda}, \underline{\mu}) = \min_{x \in D} L(x, \underline{\lambda}, \underline{\mu}) \leq L(x^*, \underline{\lambda}, \underline{\mu}) =$$

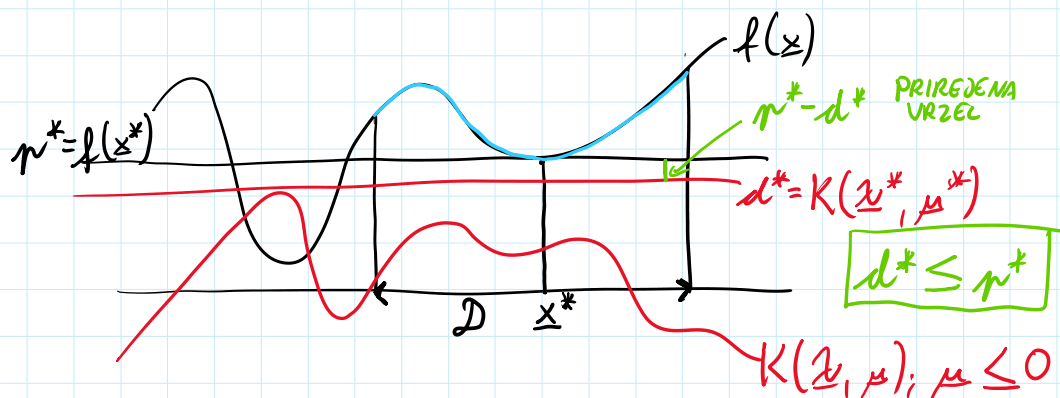
0 ≥ 0 ≥ 0

$$K(\underline{x}, \mu) = \min_{x \in D} L(x, \underline{x}, \mu) \leq L(x^*, \underline{x}, \mu) =$$

$$= f(x^*) - \underbrace{\lambda_1 g_1(x^*)}_0 - \dots - \underbrace{\lambda_m g_m(x^*)}_0 - \underbrace{\mu_1 h_1(x^*)}_{\geq 0} - \dots - \underbrace{\mu_p h_p(x^*)}_{\geq 0} \leq f(x^*) = p^*$$

$\text{ker } x^* \in D$
DODATNA PREDPOSTAVKA $\mu_i \leq 0$

Gledi: $K(\underline{x}, \mu) \leq p^*$ za vsa $\underline{x}$ in vsa $\mu \leq 0$



Primer problema: $\max K(\underline{x}, \mu)$
pri pogojih $\mu \leq 0$

Naj ima problem optimalno vrednost $v(\underline{x}^*, \mu^*)$ oz. $\max K(\underline{x}, \mu) = K(\underline{x}^*, \mu^*) = d^*$

PRIMER: $f, h: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 2x^2 - 2x + 1$$

$$h(x) = x^2 - 2x = x(x-2)$$

$$x = -\frac{b}{2a}$$

$$x^* = \frac{1}{2}$$

$$p^* = \frac{1}{2} = f(x^*)$$

$$p: \min_{\text{pogoju}} 2x^2 - 2x + 1$$

$$x^2 - 2x \leq 0 \Leftrightarrow x \in [0, 2]$$

$$L(x, \mu) = x^2(2-\mu) + x(-2+2\mu) + 1$$

$$K(\mu) = \min L(x, \mu) =$$

$$= L\left(\frac{-(-2+2\mu)}{2(2-\mu)}, \mu\right) = \frac{\mu^2 - \mu - 1}{\mu - 2}$$

$$d: \max_{\text{pogoju}} K(\mu) \quad \left. \begin{matrix} \mu \leq 0 \end{matrix} \right\} \max_{\mu \leq 0} K(\mu) = K(0) = \frac{1}{2} = d^* \quad \begin{matrix} d^* = p^* \\ \mu^* = 0 \end{matrix}$$

PRIMER 2 (LINEARNO PROGRAMIRANJE):

$$\begin{aligned} \min \quad & \vec{z}^T \vec{x} = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{pogoj} \quad & A\vec{x} \leq \vec{b} \leftarrow m \text{ linearnih pogojev} \end{aligned}$$

↓ PRIREJEN PRIMER

$$L(\vec{x}, \vec{\mu}) = \vec{z}^T \vec{x} - \vec{\mu}^T (A\vec{x} - \vec{b}) = (\vec{z} - \vec{\mu}^T A) \vec{x} - \vec{\mu}^T \vec{b}$$

$$\begin{aligned} K(\vec{\mu}) &= \inf_{\vec{x}} \{ (\vec{z} - \vec{\mu}^T A) \vec{x} + \underbrace{\vec{\mu}^T \vec{b}}_{\text{NEODVISNO OD } \vec{x}} \} = \\ &= \vec{\mu}^T \vec{b} + \underbrace{\inf_{\vec{x}} \{ (\vec{z} - \vec{\mu}^T A) \vec{x} \}}_{\text{POLJUBNO MAJHANO STEVILLO}} = \vec{\mu}^T \vec{b} + \begin{cases} 0; & \vec{z} - \vec{\mu}^T A = 0 \quad A^T \vec{\mu} = \vec{z} \\ -\infty & \text{sicer} \end{cases} \end{aligned}$$

$$K(\vec{\mu}) = \begin{cases} \vec{\mu}^T \vec{b} & \text{če } A^T \vec{\mu} = \vec{z} \\ -\infty & \text{sicer} \end{cases}$$

PRIREJEN PROBLEM:

$$\begin{aligned} \max_{\vec{\mu}} \quad & K(\vec{\mu}) \leftarrow \text{LINEARNO} \\ \text{pogoj} \quad & A^T \vec{\mu} = \vec{z} \\ & \vec{\mu} \leq 0 \end{aligned} \left. \begin{array}{l} \text{LINEARNI} \\ \text{POGOJI} \end{array} \right\} \Rightarrow \underline{d^* = p^*}$$

SE VE

Če liho $p^* - d^* \geq 0$? Je ne ve.

Če da $p^* = d^*$? Je ne ve n. p. ločno

f, h konveksne, g linearno $\Rightarrow p^* = d^*$

LIN PROGRAM:

$$\begin{aligned} \min \quad & \vec{z}^T \vec{x} \\ \text{pogoj} \quad & A\vec{x} \leq \vec{b} \\ & \vec{x} \leq 0 \end{aligned}$$

SEMIDEFINITNI PROGRAM

$$\begin{aligned} \min_{\vec{x} \in \mathbb{R}^{n \times n}} \quad & \langle C, \vec{x} \rangle_F \\ \text{pogoj} \quad & \langle A, \vec{x} \rangle_F \leq b_k \\ & \vec{x} \leq 0 \end{aligned}$$

POSLOŠITEV

KONVEKSNI PROBLEM

$$\min_{\vec{x}} f(\vec{x}) \text{ konveksno}$$

$$\text{pogoj} \quad \left. \begin{array}{l} g_i(\vec{x}) = 0 \\ h_j(\vec{x}) \leq 0 \end{array} \right\} \text{konveksna}$$

} 2 konveksna

Če je se zgodi, da $d^* = p^*$?

$$d^* = K(\vec{z}^*, \vec{\mu}^*) = \min_{\vec{x}} L(\vec{x}, \vec{z}^*, \vec{\mu}^*) \stackrel{①}{\leq} L(\vec{x}^*, \vec{z}^*, \vec{\mu}^*) = f(\vec{x}^*) - \underbrace{\vec{z}^* \vec{G}(\vec{x}^*)}_{=0} - \underbrace{\vec{\mu}^* \vec{H}(\vec{x}^*)}_{\substack{\leq 0 \leq 0 \\ \geq 0}} \stackrel{②}{\leq} f(\vec{x}^*) = p^*$$

Če je $d^* = p^*$ sta ① in ② ENAKOSTI

Ko je $d^* = p^*$ sta ① in ② ENAKOSTI

$$\textcircled{1} \min_x L(x, \lambda^*, \mu^*) = L(x^*, \lambda^*, \mu^*)$$

$\Rightarrow L(x, \lambda^*, \mu^*)$ ima v $x = x^*$ min vrednost:

$$\Rightarrow \frac{\partial L}{\partial x} \Big|_{x=x^*} = 0$$

REŠITEV ZAČETNEGA PROBLEMA JE TUDI
NIZLA VSEH ENAKOSTI $\frac{\partial L}{\partial x} = 0$

$$\textcircled{2} \mu^* \bar{H}(x^*) = 0$$

$$\underbrace{\mu_1^* h_1(x^*)}_{\geq 0} + \dots + \underbrace{\mu_p^* h_p(x^*)}_{\geq 0} = 0$$

$$\mu_i^* h_i(x^*) = 0 \text{ za } i=1 \dots p$$

PRIREJENO DOPOLNJEVANJE

Če $h_i(x^*) = 0 \Rightarrow x^*$ leži na robu območja
Če $h_i(x^*) < 0 \Rightarrow x^*$ je v notranjosti
 $\Rightarrow \mu_i^* = 0$

Če $p^* = d^*$ morata rešiti p problema (x^*) in d problema (λ^*, μ^*) reševati:

$$\left. \begin{array}{l} 1) \frac{\partial L}{\partial x}(x^*, \lambda^*, \mu^*) = 0 \\ 2) \mu_i^* h_i = 0 \text{ za } i=1 \dots p \\ 3) g_i(x^*) = 0 \\ 4) h_i(x^*) \leq 0 \\ 5) \mu_i \leq 0 \end{array} \right\} x^* \in d$$

KARUSH-KUHN-TUCKERJEV POGOJI
(KKT)

~ Zapišimo KKT za:

$$\min: (x-2)^2 + 2(y-1)^2$$

$$\text{pogoji: } x+4y \leq 3 \quad h_1(x,y) = x+4y-3 \leq 0$$

$$x=y \quad g_1(x,y) = x-y=0$$

$$\mu_i \leq 0$$

$$L(x,y, \lambda, \mu) = (x-2)^2 + 2(y-1)^2 - \lambda(x-y) - \mu(x+4y-3)$$

$$\textcircled{1} \begin{array}{l} \frac{\partial L}{\partial x} = 2(x-2) - \lambda - \mu = 0 \\ \frac{\partial L}{\partial y} = 4(y-1) + \lambda - 4\mu = 0 \end{array}$$

LAHKO ODVAJAS PO λ
!!! PO μ NE SMEŠ!!!!

$$\textcircled{2} \mu(x+4y-3) = 0$$

$$\textcircled{3} x=y$$

$$\textcircled{4} x+4y \leq 3$$

~ ~

(4.) $x+4y \leq 0$

(5.) $\mu \leq 0$

minimizacija
pri pogojih

$$\begin{aligned} f(x) \\ g_i(x) = 0 \quad i=1 \dots m \quad \vec{G}(x) = \vec{0} \\ h_j(x) \leq 0 \quad j=1 \dots p \quad \vec{H}(x) \leq \vec{0} \end{aligned}$$

$$\Downarrow$$

$$\min_{x \in D} f(x)$$

Resitev problema p^* je x^*
 $p^* = f(x^*)$

$$L(x, \lambda, \mu) = f(x) - \lambda \vec{G}(x) - \mu \vec{H}(x)$$

$$K(\lambda, \mu) = \min_x L(x, \lambda, \mu)$$

za vse λ

$$\text{za vse } \mu \leq 0 \rightarrow f(x) \geq K(\lambda, \mu)$$

za vse $x \in D$

$$p^* \geq d^*$$

PRIREDEN PROBLEM

max $K(\lambda, \mu)$
 λ, μ

Resitev je v točki (λ^*, μ^*) in
 $d^* = K(\lambda^*, \mu^*)$

pri pogojih $\mu \leq 0$

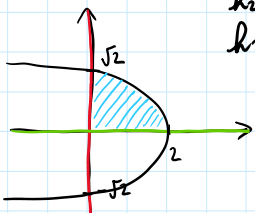
$\forall$ primeru ko $p^* = d^*$ morajo x^*, λ^*, μ^* zadoščati:

- 1) $\frac{\partial L}{\partial x}(x^*, \lambda^*, \mu^*) = 0$
- 2) $\mu_j^* h_j(x^*) = 0$
- 3) $g_i(x^*) = 0$
- 4) $h_j(x^*) \leq 0$
- 5) $\mu_i \leq 0$

Zapišimo in preizkusimo KKT za naslednji problem:

min $f(x) = -xy$
 pogoj $x+y^2 \leq 2$
 $x \geq 0$
 $y \geq 0$

$$\begin{aligned} h_1 &= x+y^2-2 \\ h_2 &= -x \\ h_3 &= -y \end{aligned}$$

KKT: ≤ 0

$$L(x, y, \mu_1, \mu_2, \mu_3) = -xy - \mu_1(x+y^2-2) - \mu_2(-x) - \mu_3(-y)$$

$$\begin{aligned} 1) \frac{\partial L}{\partial x} &= -y - \mu_1 + \mu_2 = 0 \\ \frac{\partial L}{\partial y} &= -x - 2\mu_1 y + \mu_3 = 0 \end{aligned}$$

$$\begin{aligned} 2) \mu_1(x+y^2-2) &= 0 \\ \mu_2 x &= 0 \\ \mu_3 y &= 0 \end{aligned}$$

$$H_L = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$\uparrow$
 indefinitno

$$3) \text{ NI ENAKOSTI}$$

$$\begin{aligned} 4) x+y^2-2 &\leq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

$$5) \mu_1 \leq 0 \quad \mu_2 \leq 0 \quad \mu_3 \leq 0$$

BREZ KKT:

v notrajnosti: stacionarne točke $f(x)$

3 rešeni - 3 Lagrange

3 rešene točke

LOČIMO:

a) $\mu_2 = \mu_3 = 0$: 1.1 $\rightarrow -y - \mu_1 = 0 \quad y = -\mu_1$
 1.2 $\rightarrow -x - 2\mu_1 y = 0 \quad x = -2\mu_1 y = 2\mu_1^2$
 2.1 $\rightarrow \mu_1(2\mu_1 + \mu_1^2 - 2) = 0 \quad \mu_1(3\mu_1^2 - 2) = 0$
 $\mu_1 = 0$
 $x = 0$
 $y = 0$
 $x, y, \mu_1, \mu_2, \mu_3 = (0, 0, 0, 0, 0)$
 $3\mu_1^2 - 2 = 0$
 $\mu_1^2 = \frac{2}{3}$
 $\mu_1 = \pm \sqrt{\frac{2}{3}}$
 $\mu_1 = -\sqrt{\frac{2}{3}}$
 $x = \frac{4}{3}$
 $y = \sqrt{\frac{2}{3}}$
 $(x, y, \mu_1, \mu_2, \mu_3) = (\frac{4}{3}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, 0, 0)$

b) $\mu_2 = 0 \quad \mu_3 \neq 0 \rightarrow y = 0$
 $-\mu_1 = 0$
 $-x + \mu_3 = 0$
 $x = \mu_3$
 $\mu_3 \geq 0$
 $x \geq 0$
 $\mu_3 = 0$
 c) $\mu_2 \neq 0 \quad \mu_3 = 0 \rightarrow x = 0$
 $-2\mu_1 y = 0$
 $y = 0$
 $\mu_1 = \mu_2$
 $\mu_1(-2) = 0$
 $\mu_1 = 0 = \mu_2$
 $\mu_1 = 0$
 $-y + \mu_2 = 0$
 $y = \mu_2$
 $\mu_2 \geq 0$
 $y \geq 0$

d) $\mu_2 \neq 0 \quad \mu_3 \neq 0 \Rightarrow x = y = 0$
 1.2: $\mu_3 = 0$

KONVEKSI PROBLEM:

$P_{\text{konv}} \min_x f(x)$
 pogoj: $Ax^T = b^T$ LINEARNI
 $h_j(x) \leq 0$
 f, h_j pa KONVEKSI
 Če sta 2x med seboj konveksni in hkrati PD

Naj bo $(\tilde{x}, \tilde{\lambda}, \tilde{\mu})$ nekateri KKT

TRDITEV: $\tilde{x} = x^*$ in $(\tilde{\lambda}, \tilde{\mu}) = (\lambda^*, \mu^*)$ in $p^* = d^*$

$\tilde{x}$ je rešitev P_{konv} , $(\tilde{\lambda}, \tilde{\mu})$ je rešitev D_{konv} in $p^* = d^*$

DOKAZ:

$\frac{\partial L}{\partial x}(\tilde{x}, \tilde{\lambda}, \tilde{\mu}) = 0$
 $\tilde{\mu}_j h_j(\tilde{x}) = 0$
 $A\tilde{x}^T = b^T$
 $h_j(\tilde{x}) \leq 0$
 $\tilde{\mu}_j \leq 0$
 $L(x, \lambda, \mu) = f(x) - \lambda(Ax^T - b^T) - \mu \vec{H}(x)$
 $\frac{\partial L}{\partial x}(x, \lambda, \mu) = \frac{\partial f}{\partial x}(x) - \lambda A - \mu \frac{\partial H}{\partial x}(x)$
 $0 = \frac{\partial L}{\partial x}(\tilde{x}, \tilde{\lambda}, \tilde{\mu}) = \frac{\partial f}{\partial x}(\tilde{x}) - \tilde{\lambda} A - \tilde{\mu} \frac{\partial H}{\partial x}(\tilde{x}) \leftarrow \tilde{x} \text{ je stacionarna točka } L$
 $\frac{\partial L}{\partial x^*}(x, \lambda, \mu) = \underbrace{\frac{\partial f}{\partial x}(x)}_{\substack{\text{PD} \\ H_f \text{ je PD} \\ f \text{ konveksna}}} - 0 - \underbrace{\mu_1 \frac{\partial h_1}{\partial x}(x)}_{\substack{\text{PD} \\ H_{h_1} \text{ je PD}}} - \dots - \underbrace{\mu_n \frac{\partial h_n}{\partial x}(x)}_{\substack{\text{PD} \\ H_{h_n} \text{ je PD}}} \leftarrow \text{VSOTA POZITIVNO DEFINITIVNIH MATRIK}$
 $\tilde{x}^T (A + \beta_1 B_1 + \dots + \beta_n B_n) \tilde{x} \geq 0$

$\tilde{x}$ je lokalni minimum $L \xrightarrow{\text{konv}} \tilde{x}$ je minimum L

$K(\tilde{\lambda}, \tilde{\mu}) = \min_x L(x, \tilde{\lambda}, \tilde{\mu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\mu}) = f(\tilde{x}) - \tilde{\lambda} \vec{G}(\tilde{x}) - \tilde{\mu} \vec{H}(\tilde{x})$
 $= f(\tilde{x})$

$$K(\tilde{x}, \tilde{\mu}) = \min_{\lambda} L(\tilde{x}, \tilde{\mu}, \lambda) = L(\tilde{x}, \tilde{\mu}, \lambda^*) = f(\tilde{x}) \quad \text{dual value}$$

$$K(\tilde{x}, \tilde{\mu}) = f(\tilde{x}) \Rightarrow \min_{\tilde{x}} f(\tilde{x}) = \max_{\tilde{\mu} \leq 0} K(\tilde{x}, \tilde{\mu})$$

$$\underline{d^* = p^*}$$