Ime in priimek

Vpisna številka

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2. kolokvij iz Matematike 1

18. januar 2024

Čas pisanja: **90 minut**. *Dovoljena je uporaba dveh listov velikosti A4 za pomoč. Prepisovanje, pogovarjanje in uporaba knjig, zapiskov, pametnega telefona in ostalih elektronskih naprav je* **strogo** *prepovedano.*

1. naloga (30 točk)

Telo $D \subseteq \mathbb{R}^3$ je presek pokončnega valja, ki je definiran z neenačbo

$$x^2 + y^2 \leq 1,$$

in vodoravnega valja, ki je definiran z neenačbo

$$x^2 + z^2 \leq 1.$$

Izračunaj maso tega telesa, če je funkcija gostote dana z

$$\rho(x,y,z) = \frac{1}{\sqrt{1-x^2}}.$$

Namig: Upoštevaj, da je projekcija preseka teh dveh valjev na xy-ravnino enotski krog. Uporabi cilindrične koordinate.

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

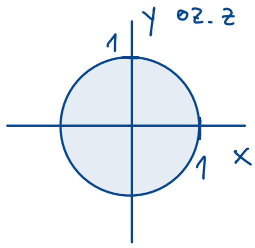
$\det(J_{\vec{F}}) = r$

$x^2+y^2 \leq 1 \;\ldots\; r^2 \leq 1 \;,\; r \leq 1 \;,\; r \in [0,1]$

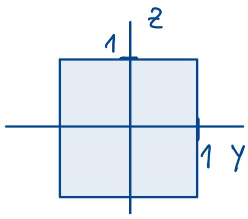
$x^2+z^2 \leq 1 \;\ldots\; r^2\cos^2\varphi + z^2 \leq 1 \;\ldots\; z^2 \leq 1 - r^2\cos^2\varphi$

$\ldots\; -\sqrt{1-r^2\cos^2\varphi} \leq z \leq \sqrt{1-r^2\cos^2\varphi}$

Projekciji telesa na xy- in xz-ravnino:



Projekcija telesa na yz-ravnino:



$$m = \iiint_D \rho(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi}\left(\int_0^1\left(\int_{-\sqrt{1-r^2\cos^2\varphi}}^{\sqrt{1-r^2\cos^2\varphi}} \frac{r}{\sqrt{1-r^2\cos^2\varphi}}\,dz\right)dr\right)d\varphi =$$

$$= \int_0^{2\pi}\left(\int_0^1\left(\frac{2r\cancel{\sqrt{1-r^2\cos^2\varphi}}}{\cancel{\sqrt{1-r^2\cos^2\varphi}}}\right)dr\right)d\varphi = \underbrace{\left(\int_0^{2\pi} d\varphi\right)}_{2\pi}\underbrace{\left(\int_0^1 2r\,dr\right)}_{1} = \underline{\underline{2\pi}}.$$

2. naloga (30 točk)

Funkcija dveh spremenljivk h ima predpis

$$h(x,y) = (x+y)^4 - 32xy.$$

Poišči vse stacionarne točke funkcije h in jih klasificiraj.

$$\operatorname{grad} h = \left[\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right] = [0,0],$$

$$\left.\begin{aligned} \frac{\partial h}{\partial x} &= 4(x+y)^3 - 32y = 0 \\ \frac{\partial h}{\partial y} &= 4(x+y)^3 - 32x = 0 \end{aligned}\right\} -$$

$$-32y + 32x = 0 \ \dots \ x = y$$

$$4\cdot(2y)^3 - 32y = 0$$

$$32y\,(y^2-1) = 0$$

$$y_1 = -1,\ y_2 = 0,\ y_3 = 1$$

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1$$

Tri stacionarne točke: $\underline{\underline{T_1(-1,-1),\ T_2(0,0),\ T_3(1,1)}}$.

Hessejeva matrika:

$$H_h = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial y \partial x} \\ \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12(x+y)^2 & 12(x+y)^2 - 32 \\ 12(x+y)^2 - 32 & 12(x+y)^2 \end{bmatrix}.$$

$$H_h(T_1) = H_h(-1,-1) = \begin{bmatrix} 48 & 16 \\ 16 & 48 \end{bmatrix},$$

$48 > 0$, $\det(H_h(T_1)) > 0$, mat. je PD po Sylvestrovem kriteriju, $\underline{\underline{T_1 \text{ je lokalni minimum}}}$

$$H_h(T_2) = H_h(0,0) = \begin{bmatrix} 0 & -32 \\ -32 & 0 \end{bmatrix},$$

$\det(H_h(T_2)) < 0$, tj. lastni vrednosti $H_h(T_2)$ sta nasprotno predznačeni, $\underline{\underline{T_2 \text{ je sedlo}}}$

$H_h(T_3) = H_h(T_1)$, $\underline{\underline{T_1 \text{ je lokalni minimum}}}$

3. naloga (35 točk)

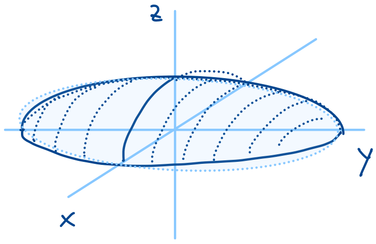
Naj bo

$$f(x,y,z) = x + 2y + z.$$

Poišči najmanjšo vrednost funkcije f pri pogojih

$$x^2 + 4y^2 + z^2 \leq 4 \text{ in } z \geq 0.$$

Pogoja določata del elipsoida v polprostoru $z \geq 0$.



Kandidati za ekstreme bodo v notranjosti in na robu, ki je sestavljen iz ravnine $z=0$, elipsoida $x^2+4y^2+z^2=4$ ter njunega preseka.

- notranjost: $\text{grad } f = [1,2,1] \neq [0,0,0]$, ekstrema v notranjosti ni.
- rob, elipsoid: $L(x,y,z,\lambda) = x+2y+z-\lambda(x^2+4y^2+z^2-4)$

$$\left.\begin{aligned}
\frac{\partial L}{\partial x} &= 1-2\lambda x = 0 \ \dots \ x = \frac{1}{2\lambda} \\
\frac{\partial L}{\partial y} &= 2-8\lambda y = 0 \ \dots \ 2y = \frac{1}{2\lambda} \\
\frac{\partial L}{\partial z} &= 1-2\lambda z = 0 \ \dots \ z = \frac{1}{2\lambda}
\end{aligned}\right\} \ x = 2y = z \ \dots \ 4y^2+4y^2+4y^2 = 4$$

$$\frac{\partial L}{\partial \lambda} = -(x^2+4y^2+z^2-4) = 0$$

$$y^2 = \frac{1}{3} \ \dots \ y = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow x = z = \pm\frac{2}{\sqrt{3}}$$

Ker mora biti $z \geq 0$, imamo le $\underline{T_1\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$

- rob, ravnina $z=0$: Poenostavimo, $f(x,y,0) = x+2y$, ta nima stac. točk.
- rob, presek elipsoida in ravnine, tj. elipsa z en. $x^2+4y^2=4$:

$$L(x,y,\lambda) = f(x,y,0) - \lambda(x^2+4y^2-4) = x+2y-\lambda(x^2+4y^2-4)$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial x} &= 1-2x\lambda = 0 \ \dots \ x = \frac{1}{2\lambda} \\
\frac{\partial L}{\partial y} &= 2-8y\lambda = 0 \ \dots \ 2y = \frac{1}{2\lambda}
\end{aligned}\right\} \ x = 2y \ \dots \ 4y^2+4y^2 = 4 \ \dots \ y^2 = \frac{1}{2} \ \dots \ y = \pm\frac{1}{\sqrt{2}}$$

$$\frac{\partial L}{\partial \lambda} = -(x^2+4y^2-4) = 0$$

$$x = \pm\sqrt{2}$$

$$z = 0$$

Dobimo še dva kandidata $\underline{T_{2,3}\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, 0\right)}$

(x,y,z)	T_1	T_2	T_3
$f(x,y,z)$	$2\sqrt{3}$	$2\sqrt{2}$	$-2\sqrt{2}$

najmanjša vrednost f pri danih pogojih.